Hypothesis Testing with Two Samples

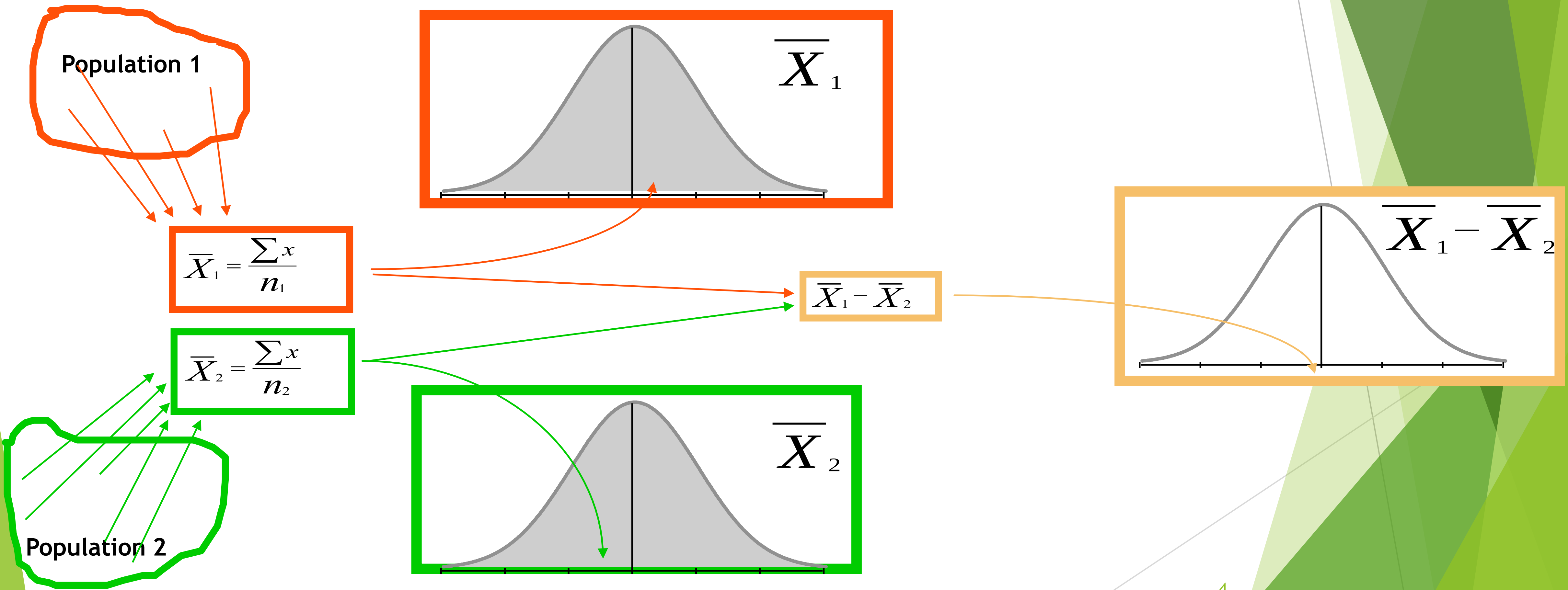
Learning Objectives

- Test hypotheses about the difference in two population means using data from large independent samples.
- Test hypotheses about the difference in two population means using data from small independent samples when the populations are normally distributed.

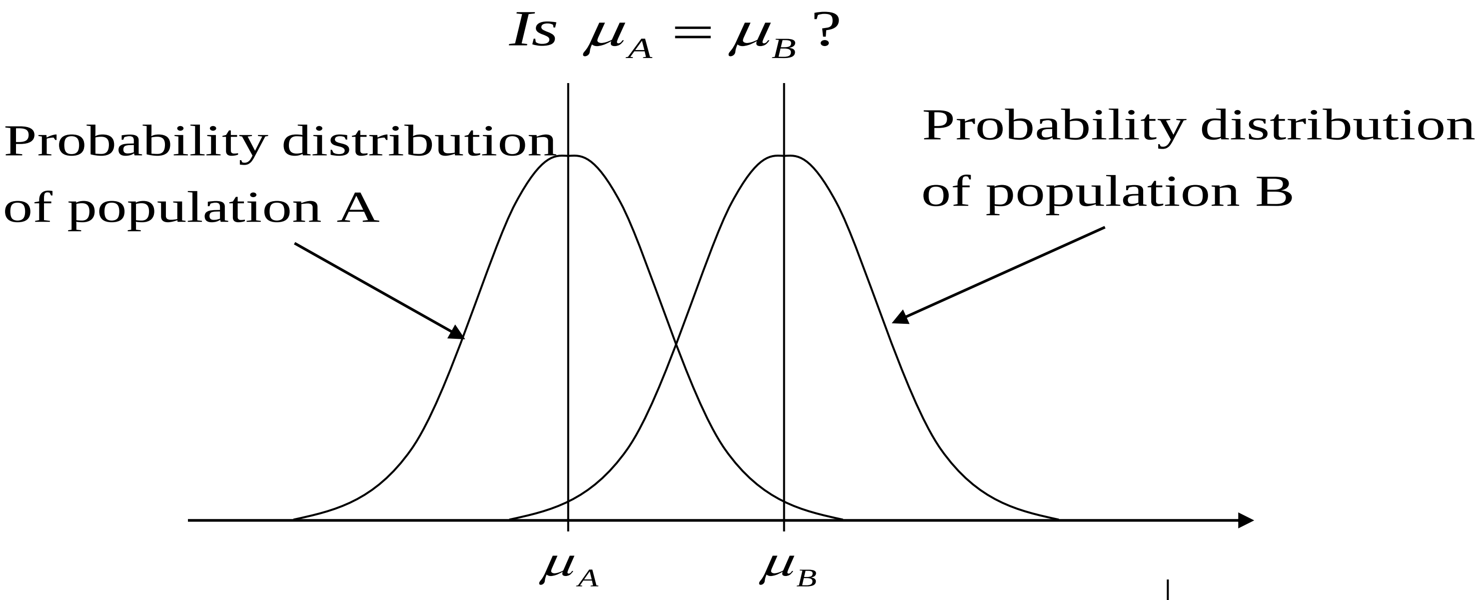
Learning Objectives, *continued*

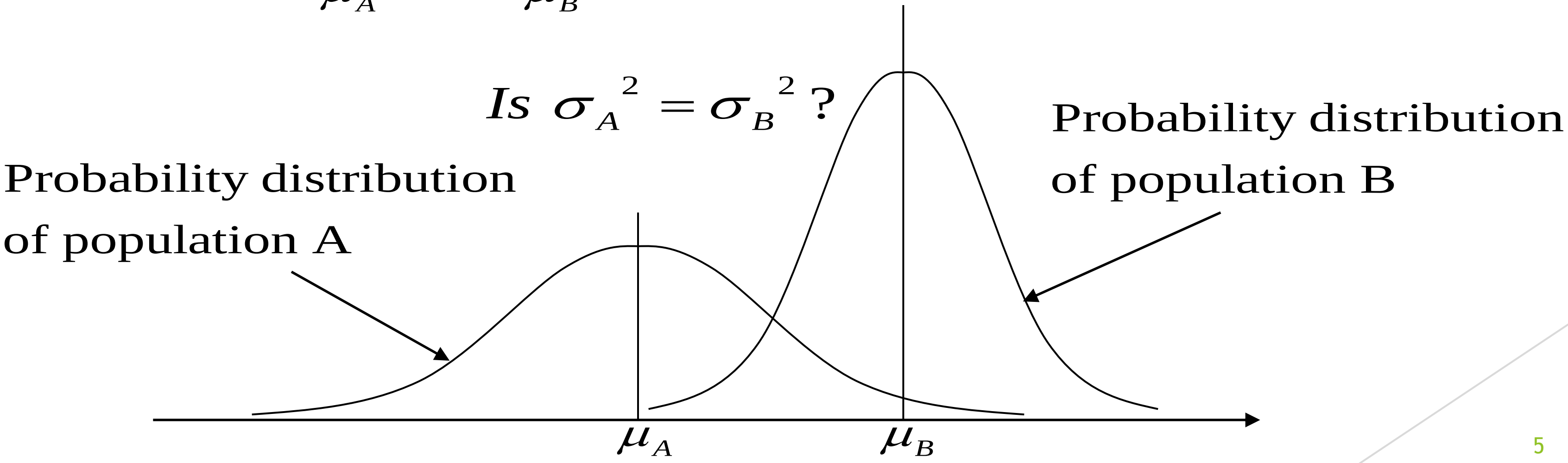
- Test hypotheses about the mean difference in two related populations when the populations are normally distributed.
- Test hypotheses about the differences in one (two) population proportion(s).

Hypothesis Testing about the Difference in Two Sample Means



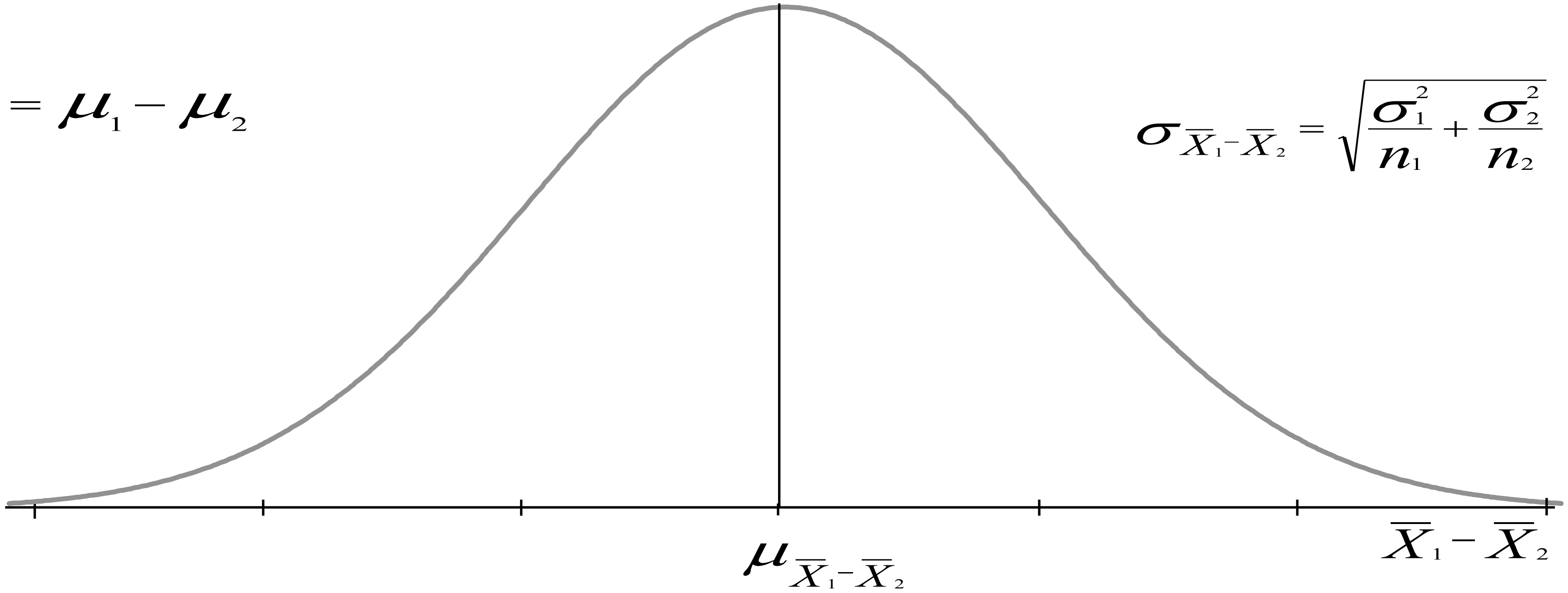
Hypothesis Testing about the Difference in Two Sample Means





Hypothesis Testing about the Difference in Two Sample Means

$$\mu_{\bar{X}_1-\bar{X}_2} = \mu_1 - \mu_2$$



$$\sigma_{\bar{X}_1-\bar{X}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Z Formula for the Difference in Two Sample Means for $n_1 \geq 30$, $n_2 \geq 30$, and Independent Samples , Unequal Variances

$$Z = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

The *t* Test for Differences in Population Means

- Each of the two populations is normally distributed.
- The two samples are independent.
- At least one of the samples is small, $n < 30$.
- The values of the population variances are unknown.

Test of (μ_1 – μ_2), Unequal Variances, Independent Samples, $n_1 < 30, n_2 < 30$

- Test Statistic

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

$$\text{where} \quad df = \frac{\left[(s_1^2/n_1) + (s_2^2/n_2)\right]^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$$

Example: Hypothesis Testing for Differences Between Means (Part 1)

Computer Analysts		
24.10	25.00	24.25
23.75	22.70	21.75
24.25	21.30	22.00
22.00	22.55	18.00
23.50	23.25	23.50
22.80	22.10	22.70
24.00	24.25	21.50
23.85	23.50	23.80
24.20	22.75	25.60
22.90	23.80	24.10
23.20		
23.55		

$$n_1 = 32$$
$$\overline{X}_1 = 23.14$$
$$S_1 = 1.373$$
$$S_1^2 = 1.885$$

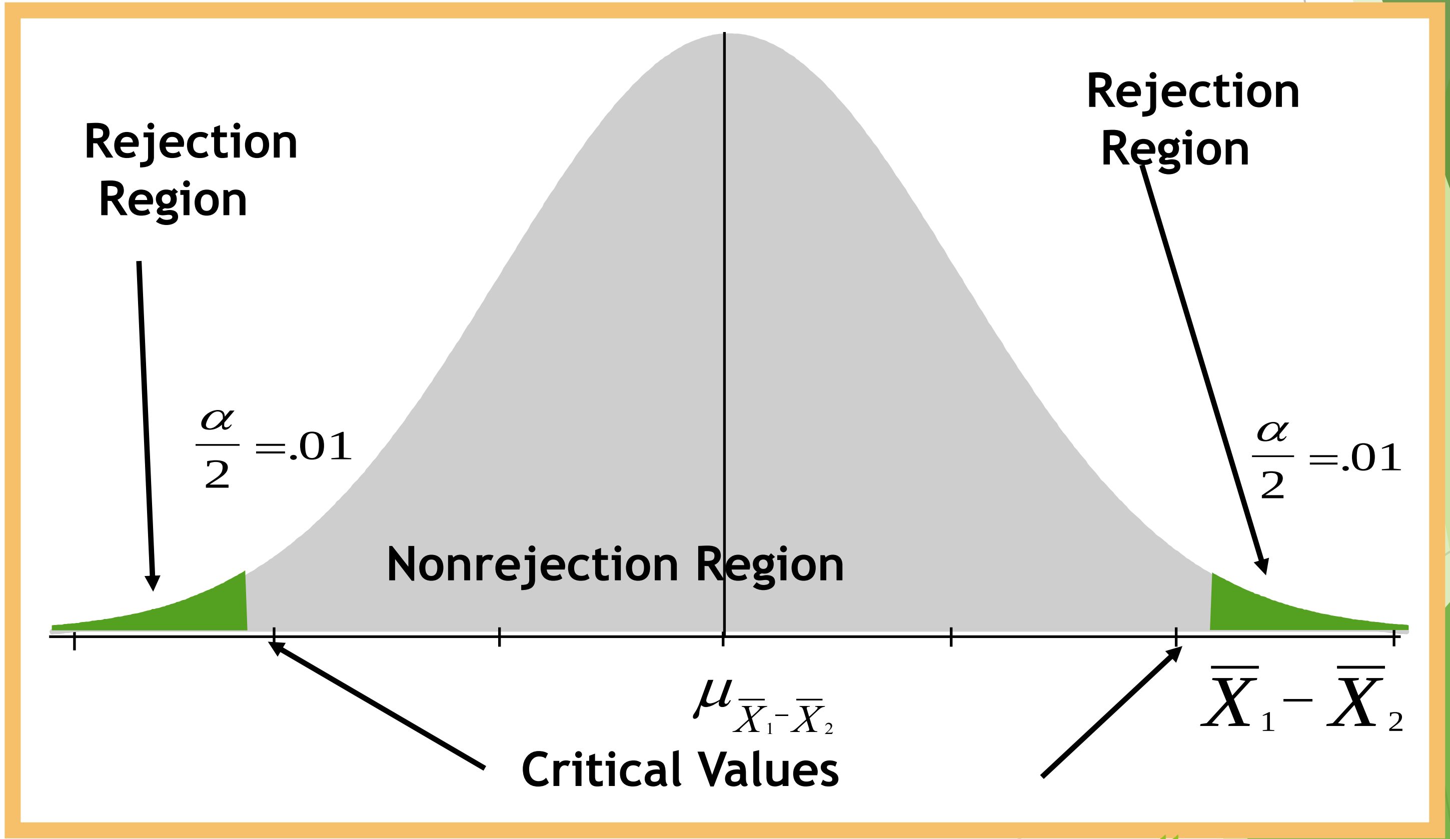
$$n_2 = 34$$
$$\overline{X}_2 = 21.99$$
$$S_2 = 1.403$$
$$S_2^2 = 1.968$$

Registered Nurses		
20.75	23.30	22.75
23.80	24.00	23.00
22.00	21.75	21.25
21.85	21.50	20.00
24.16	20.40	21.75
21.10	23.25	20.50
23.75	19.50	22.60
22.50	21.75	21.70
25.00	20.80	20.75
22.70	20.25	22.50
23.25	22.45	
21.90	19.10	

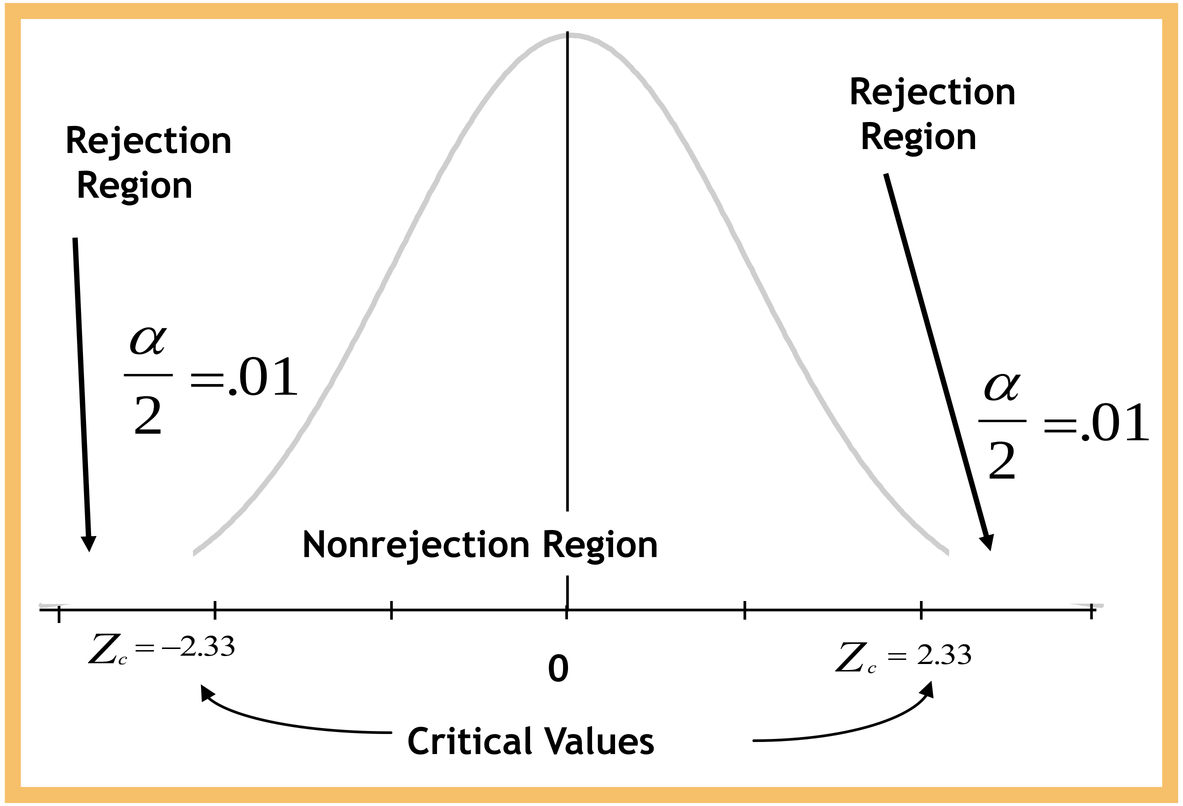
Example: Hypothesis Testing for Differences Between Means (Part 2)

$$H_o: \mu_1 - \mu_2 = 0$$

$$H_a: \mu_1 - \mu_2 \neq 0$$



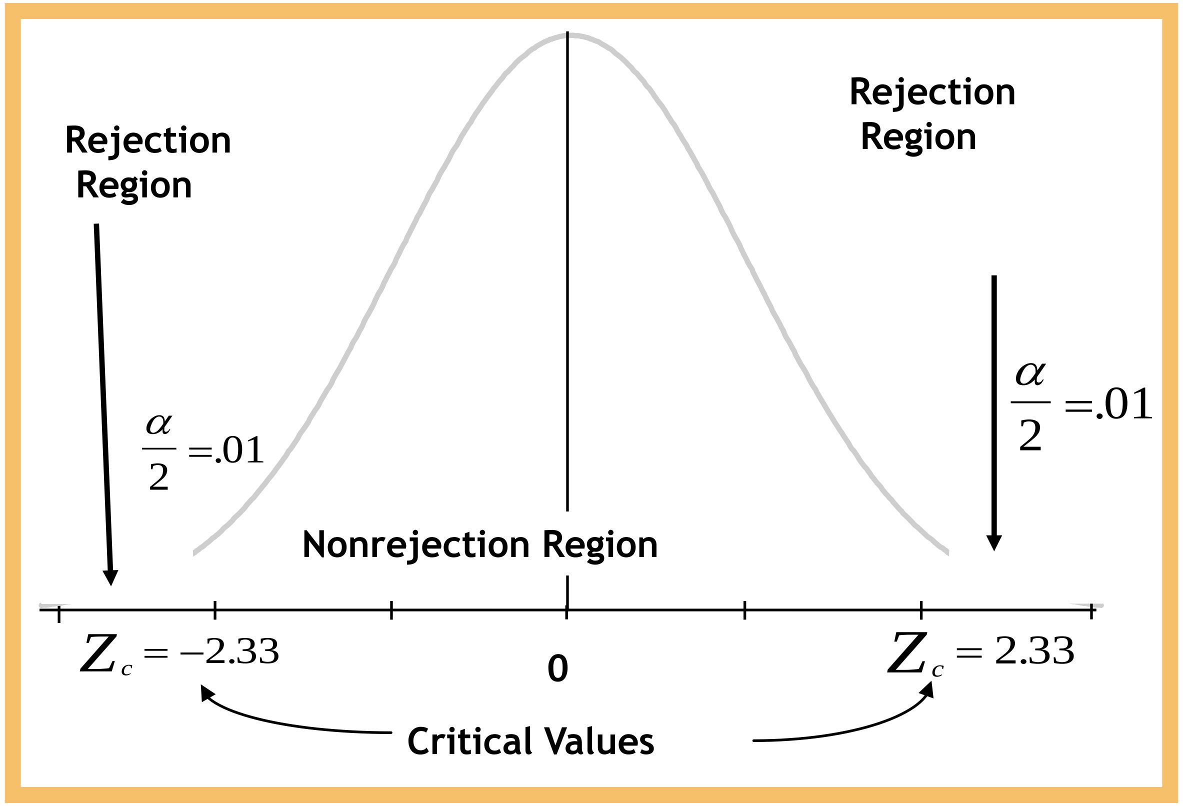
Example: Hypothesis Testing for Differences Between Means (Part 3)



If $Z < -2.33$ or $Z > 2.33$, reject H_o.

If $-2.33 \leq Z \leq 2.33$, do not reject H_o.

Example: Hypothesis Testing for Differences between Means (Part 4)



If Z < -2.33 or Z > 2.33, reject Ho.

If -2.33 ≤ Z ≤ 2.33, do not reject Ho.

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

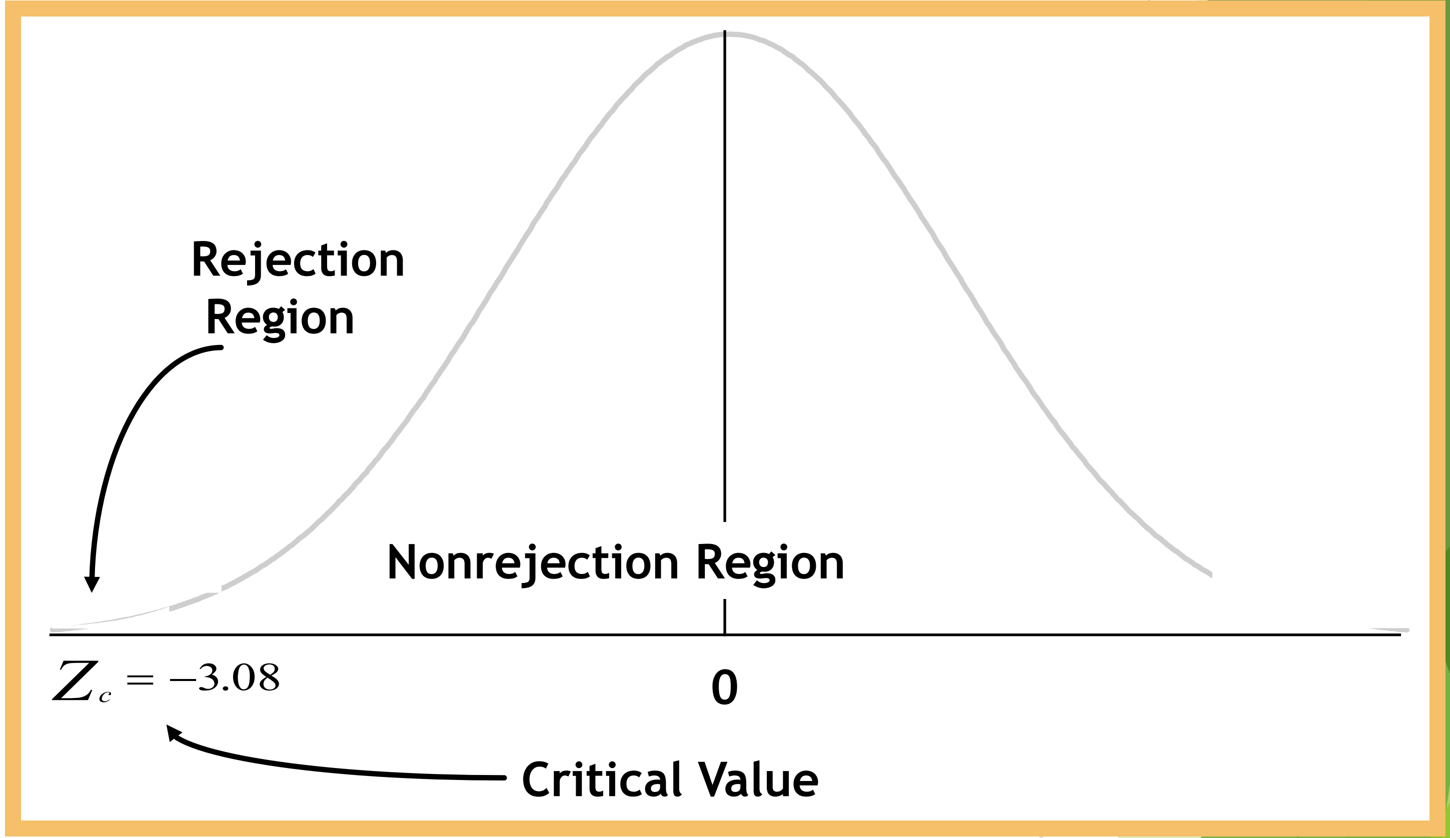
$$= \frac{(23.14 - 21.99) - (0)}{\sqrt{\frac{1.885}{32} + \frac{1.968}{34}}} = 3.36$$

Since Z = 3.36 > 2.33, reject Ho.

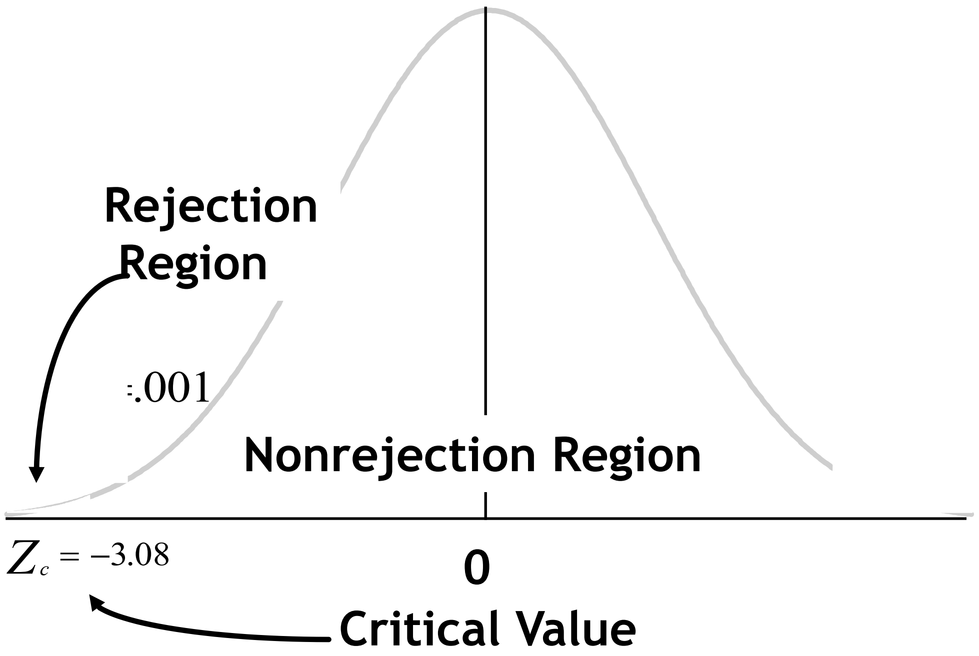
Demonstration Problem 1 (Part 1)

$$H_o: \mu_1 - \mu_2 = 0$$

$$H_a: \mu_1 - \mu_2 < 0$$



Demonstration Problem 1 (Part 2)



If Z < -3.08, reject Ho.

If Z ≥ −3.08, do not reject Ho.

Women

$\bar{X}_1 = \$3,343$

$S_1 = \$1,226$

$n_1 = 87$

Men

$\bar{X}_2 = \$5,568$

$S_2 = \$1,716$

$n_2 = 76$

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{(3343 - 5568) - (0)}{\sqrt{\frac{1226^2}{87} + \frac{1716^2}{76}}} = -9.40$$

Since Z = -9.40 < -3.08, reject Ho.

Test of (μ_1 - μ_2), Equal Variances, Independent Samples, $n_1 < 30$, $n_2 < 30$

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2(n_1 - 1) + S_2^2(n_2 - 1)}{n_1 + n_2 - 2}} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Hernandez Manufacturing Company (Part 1)

Training Method A		
56	51	45
47	52	43
42	53	52
50	42	48
47	44	44

Training Method B		
59	57	53
52	56	65
53	55	53
54	64	57

$$n_1 = 15$$
$$\overline{X}_1 = 47.73$$
$$S_1^2 = 19.495$$

$$n_2 = 12$$
$$\overline{X}_2 = 56.5$$
$$S_2^2 = 18.273$$

Hernandez Manufacturing Company (Part 2)

$$H_o: \mu_1 - \mu_2 = 0$$

$$H_a: \mu_1 - \mu_2 \neq 0$$

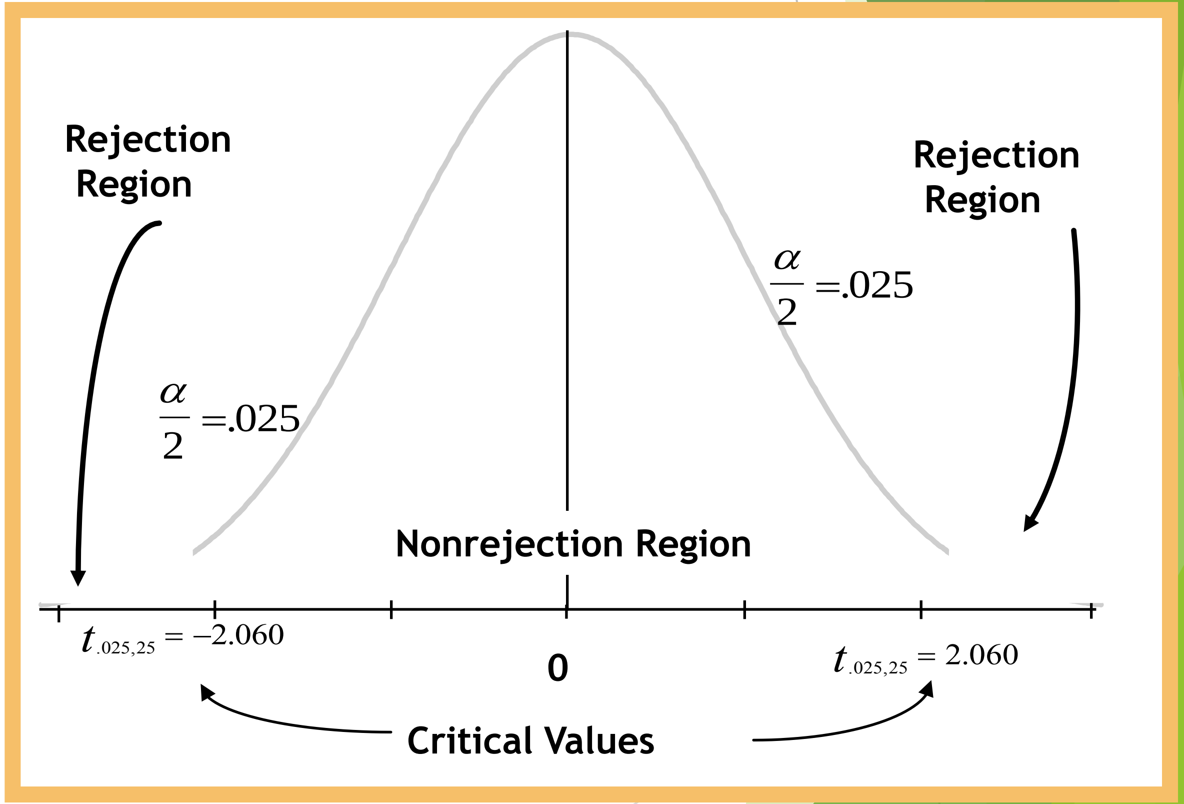
$$\frac{\alpha}{2} = \frac{.05}{2} = .025$$

$$df = n_1 + n_2 - 2 = 15 - 12 - 2 = 25$$

$$t_{0.25,25} = 2.060$$

If $t < -2.060$ or $t > 2.060$, reject H_o.

If $-2.060 \leq t \leq 2.060$, do not reject H_o.



Hernandez Manufacturing Company (Part 3)

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2(n_1-1) + S_2^2(n_2-1)}{n_1+n_2-2}}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

$$= \frac{(47.73 - 56.50) - 0}{\sqrt{\frac{(19.495)(14) + (18.273)(11)}{15+12-2}}\sqrt{\frac{1}{15}+\frac{1}{12}}}$$

$$= -5.20$$

If $t < -2.060$ or $t > 2.060$, reject H_o.

If $-2.060 \le t \le 2.060$, do not reject H_o.

Since $t = -5.20 < -2.060$, reject H_o.

Confidence Interval for (μ1 – μ2)

- The (1 – a)% confidence interval for the difference in two means:
  - Equal-variances t-interval

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \cdot \sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

  - Unequal-variances t-interval

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Confidence Interval for (μ_1 – μ_2)

- The (1 – a)% confidence interval for the difference in two means:
  - Known-variances *z*-interval

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Dependent Samples

- Before and After Measurements on the same individual
- Studies of twins
- Studies of spouses

Individual	Before	After	d
1	32	39	-7
2	11	15	-4
3	21	35	-14
4	17	13	4
5	30	41	-11
6	38	39	-1

Formulas for Dependent Samples

$$t = \frac{\bar{d} - D}{\frac{S_d}{\sqrt{n}}}$$

$$df = n - 1$$

$n =$ number of pairs

d = sample difference in pairs

D = mean population difference

S_d = standard deviation of sample difference

$\bar{d}$ = mean sample difference

$$\bar{d} = \frac{\sum d}{n}$$

$$S_d = \sqrt{\frac{\sum (d - \bar{d})^2}{n-1}} = \sqrt{\frac{\sum d^2 - \frac{(\sum d)^2}{n}}{n-1}}$$

$$\bar{d} \pm t_{\alpha/2,\nu}\left(\frac{s_d}{\sqrt{n}}\right)$$

Example - Evaluation of Transdermal Contraceptive Patch In Adolescents

- Subjects: Adolescent Females on O.C. who then received Ortho Evra Patch
- Response: 5-point scores on ease of use for each type of contraception (1=Strongly Agree)
- Data: d_i = difference (O.C.-EVRA) for subject *i*
- Summary Statistics:

$$\bar{d} = 1.77 \quad s_d = 1.48 \quad n = 13$$

Example - Evaluation of Transdermal Contraceptive Patch In Adolescents

- 2-sided test for differences (α=0.05)
- $H_0: \mu_D = 0$ $H_A: \mu_D \neq 0$

$$TS: t_{obs} = \frac{1.77}{1.48 / \sqrt{13}} = \frac{1.77}{0.41} = 4.31$$

$$RR: |t_{obs}| \geq t_{.025,12} = 2.179$$

$$95\% CI: \quad 1.77 \pm 2.179(0.41) \equiv 1.77 \pm 0.89 \equiv (0.88, 2.66)$$

Conclude Mean Scores are higher for O.C., girls find the Patch easier to use (low scores are better)

Hypothesis Testing for Population Proportion

z Test Statistic for a Proportion

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

$$\hat{p} = \frac{\text{number of successes in the random sample}}{\text{number of observations in the random sample}}$$

= sample estimator of the population proportion

p_0 = hypothesized population proportion

n = size of the sample used to compute

When we can use Z

When n is large, the normal distribution can be used to approximate the probability of r or more successes. The approximation is excellent if

(a) the population is at least 10 times larger than the sample and

(b) $np_0 > 15$ and $n(1 - p_0) > 15$, where p_0 is the hypothesized proportion.

Statistical Hypotheses for a Proportion

$$H_0 : p = p_0$$
$$H_1 : p \neq p_0$$

$$H_0 : p \geq p_0$$
$$H_1 : p < p_0$$

$$H_0 : p \leq p_0$$
$$H_1 : p > p_0$$

Confidence Interval for a Population Proportion, p

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Computational Example (part 1)

Student Congress believes that the proportion of parking tickets issued by the campus police this year is greater than last year. Last year the proportion was $p_0 = .21$.

they obtained a random sample of $n = 200$ students and found that the proportion who received tickets this year was .27.

Computational Example (part 2)

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}} = \frac{.27 - .21}{\sqrt{\frac{.21(1 - .21)}{200}}} = 2.08$$

$z_{.05} = 1.645$

The null hypothesis can be rejected; the campus police are issuing more tickets this year.

Computational Example (part 3)

Two-sided 100(1 – .05)% = 95% confidence interval

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$.27 - 1.96\sqrt{\frac{.27(1-.27)}{200}} < p < .27 + 1.96\sqrt{\frac{.27(1-.27)}{200}}$$

$$.208 < p < .332$$

Hypothesis for large independent samples

For testing **equality** of the two proportions only

H_o: $(p_1 - p_2) = 0$

H_A: $(p_1 - p_2) > 0$ (upper-tail)

$(p_1 - p_2) < 0$ (lower-tail)

$(p_1 - p_2) = 0$ (two-tail, use CI approach)

Test statistic for large independent samples

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim \text{standard normal, i.e. } N(0,1)$$

where,

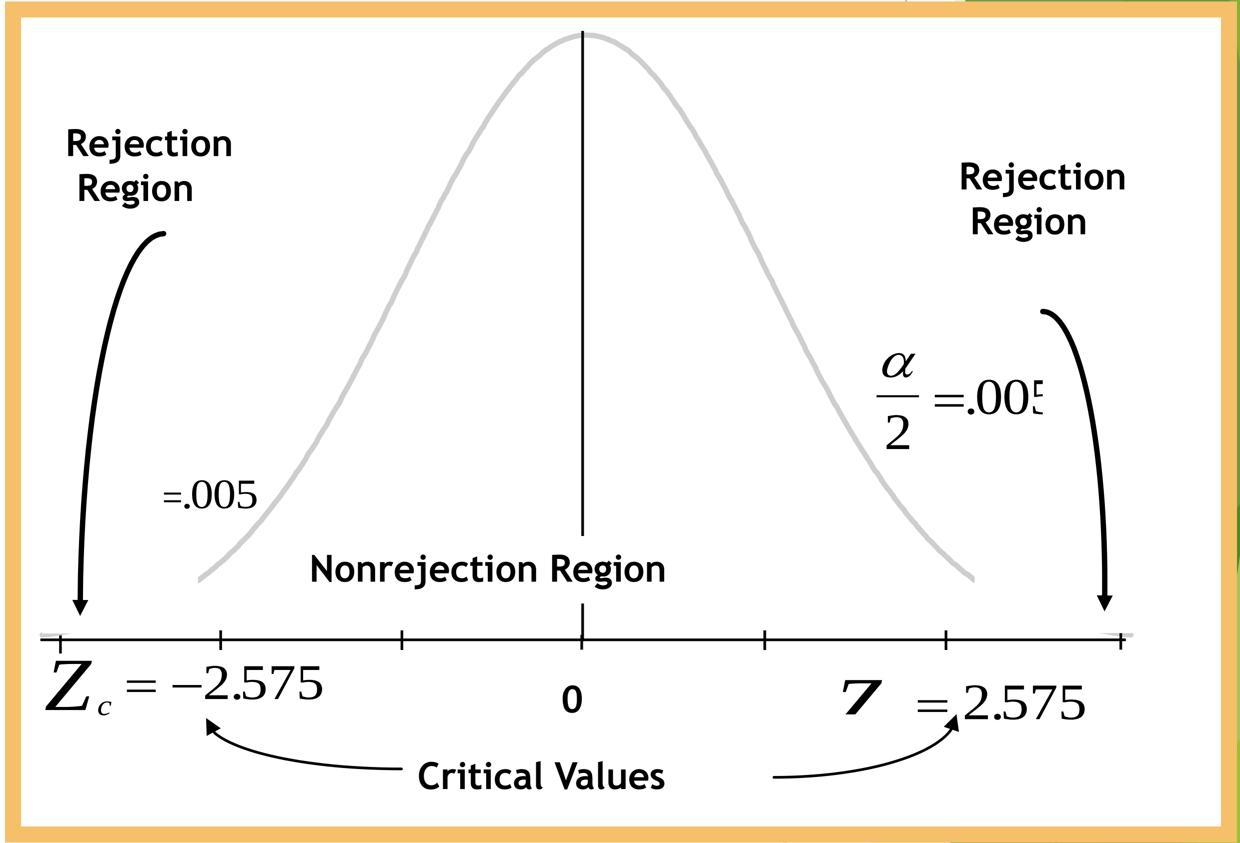
$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2} \quad \text{and} \quad \bar{q} = 1 - \bar{p}$$

Testing the Difference in Population Proportions: Example

$$H_o: P_1 - P_2 = 0$$
$$H_a: P_1 - P_2 \neq 0$$

$$\frac{\alpha}{2} = \frac{.01}{2} = .005$$
$$Z_{.005} = 2.575$$

If Z < -2.575 or Z > 2.575, reject Ho.
If -2.575 ≤ Z ≤ 2.575, do not reject Ho.



Example, *continued*

$$n_1 = 100$$
$$X_1 = 24$$
$$\hat{p}_1 = \frac{24}{100} = .24$$

$$n_2 = 95$$
$$X_2 = 39$$
$$\hat{p}_2 = \frac{39}{95} = .41$$

$$\overline{P} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{24 + 39}{100 + 95} = .323$$

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (P_1 - P_2)}{\sqrt{(\overline{P} \cdot \overline{Q})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{(.24 - .41) - (0)}{\sqrt{(.323)(.677)\left(\frac{1}{100} + \frac{1}{95}\right)}} = \frac{-.17}{.067} = -2.54$$

Since $-2.575 \leq Z = -2.54 \leq 2.575$, do not reject H_0.

hypothesis for Large Independent Samples

For testing to see if difference is at least D

H_o: $(p_1 - p_2) = D$

H_A: $(p_1 - p_2) > D$ (upper-tail)

$(p_1 - p_2) < D$ (lower-tail)

Provided $n_1p_1 \geq 10$ & $n_1q_1 \geq 10$

$n_2p_2 \geq 10$ & $n_2q_2 \geq 10$

Test Statistic for Large Independent Samples

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - \Delta}{\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}} \sim \text{standard normal, i.e. } N(0,1)$$

where,

$\hat{q}_1 = 1 - \hat{p}_1$, $\hat{q}_2 = 1 - \hat{p}_2$ and $\Delta = (p_1 - p_2) > 0$

confidence interval for (p_1 - p_2)

$$(estimate) \pm (z - value) SE(estimate)$$

$$(\hat{p}_1 - \hat{p}_2) \pm (z - value)\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$$

where,

$\hat{q}_1 = 1 - \hat{p}_1$ and $\hat{q}_2 = 1 - \hat{p}_2$

Example: Nutrition Education for Pregnant Teens

Here we are interested in determining if pregnant teens who receive the nutrition education have a lower prevalence of low birth weight infants, but we are not necessarily looking for a certain size (D) for that difference. Let,

p_E = proportion of babies with low birth weight born to teens who underwent nutrition education.

p_N = proportion of babies with low birth weight born to teens who did not receive nutrition education.

Example (part 2)

State Hypotheses

H_o: $p_E = p_N$ or equivalently $(p_E - p_N) = 0$
H_A: $p_E < p_N$ or equivalently $(p_E - p_N) < 0$

Determine Test Criteria

a) Choose a = .05 (we could use something else)

b) From the CDC website we find that around 9% of infants born in the U.S. are classified as having low birth weights. For teen mothers that percentage is probably higher but smaller p's require larger samples, thus we will use p = .09 to check sample size considerations. Here $n_1 = 314$ and $n_2 = 316$ so …

Example (part 3)

$n_1p_1 = 28$, $n_1q_1 = 286$, $n_2p_2 = 28$, $n_2q_2 = 288$ (i.e. samples are LARGE)

THUS WE USE LARGE SAMPLE TEST FOR COMPARING POPULATION PROPORTIONS ASSUMING EQUALITY UNDER THE NULL, i.e. $\Delta = 0$.

Example (part 4)

In the study, 23 of the 314 teen mothers receiving nutrition education had low birth weight babies compared to 39 of the 316 mothers in the non-instruction group.

Sample Proportion Calculations

$$\hat{p}_1 = \frac{X_1}{n_1} = \frac{23}{314} = .0732$$

$$\hat{p}_2 = \frac{X_2}{n_2} = \frac{39}{316} = .1234$$

$$\overline{p} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{23+39}{314+316} = .0984$$

$$\overline{q} = 1 - \overline{p} = .9016$$

Example (part 5)

z-test statistic shown below.

Test Statistic Calculations

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim \mathrm{N}(0,1)$$

where,

$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2} \quad \text{and} \quad \bar{q} = 1 - \bar{p}$$

$$\hat{p}_1 = .0732, \quad \hat{p}_2 = .1234$$

$$\bar{p} = .0984, \quad \bar{q} = .9016$$

$$z = \frac{.0732 - .1234}{\sqrt{(.0984)(.9016)\left(\frac{1}{314} + \frac{1}{316}\right)}}$$

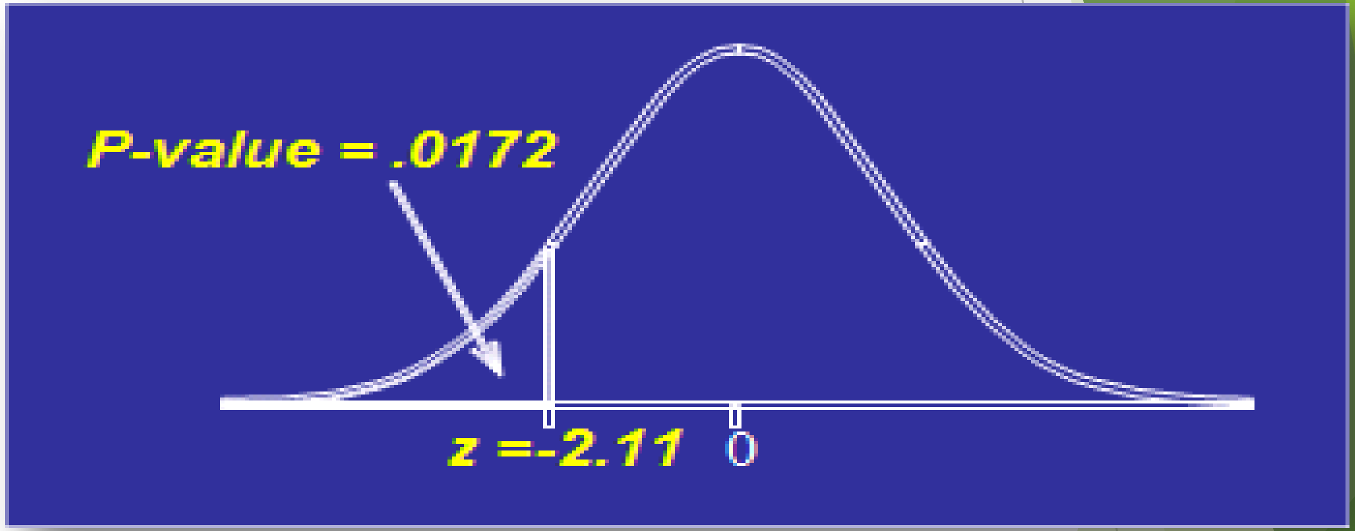
$$z = -2.12$$

Example (part 6)

Compute p-value and make decision

From standard normal table or computer

$$P(Z < -2.11) = .0172$$



The probability that chance variation alone would produce an observed proportion for education group this small or smaller when compared to the non-instruction group is 1.72%. **Thus we have evidence to suggest that the proportion of low birth weight babies born to teen mothers in education group is smaller than that for the non-instruction group (p = .0172).**

Example (part 7)

95% CI for Difference in Proportions

$$(estimate) \pm (z - value) SE(estimate)$$

$$(\hat{p}_1 - \hat{p}_2) \pm (z - value)\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$$

where,

$$\hat{q}_1 = 1 - \hat{p}_1 \quad \text{and} \quad \hat{q}_2 = 1 - \hat{p}_2$$

$$(.0732 - .1234) \pm 1.96\sqrt{\frac{(.0732)(.9268)}{314} + \frac{(.1234)(.8766)}{316}}$$

$$= -.0502 \pm (1.96)(.0236)$$

$$= -.0502 \pm .0463$$

$$= (-.0965, -.0039) \quad \text{or} \quad (\text{-}9.65\%, \text{-}.39\%)$$

Example (part 8)

- 95% CI for (p_E - p_N) = (-.0985, -.0039)
  or (- 9.85%, -.39 %)

<u>One potential interpretation of CI:</u>

We estimate that the percentage of low birth weight babies born to teen mothers who participate in a nutrition education program is between .39 and 9.85 percentage points smaller than that for teen mothers who are not given this instruction.

Example (part 8)

- 95% CI for (p_E - p_N) = (-.0985, -.0039)
  or (- 9.85%, -.39 %)

  <u>Another potential interpretation of CI:</u>

For pregnant teens participating in the nutrition education program we estimate that the prevalence of low birth weight is between .39 and 9.85 percentage points smaller than that for teen mothers receiving no such education.