Confidence Interval Estimation

Objectives

- **In this chapter, you learn:**
- To construct and interpret confidence interval estimates for the population mean and the population proportion.
- To determine the sample size necessary to develop a confidence interval for the population mean or population proportion.

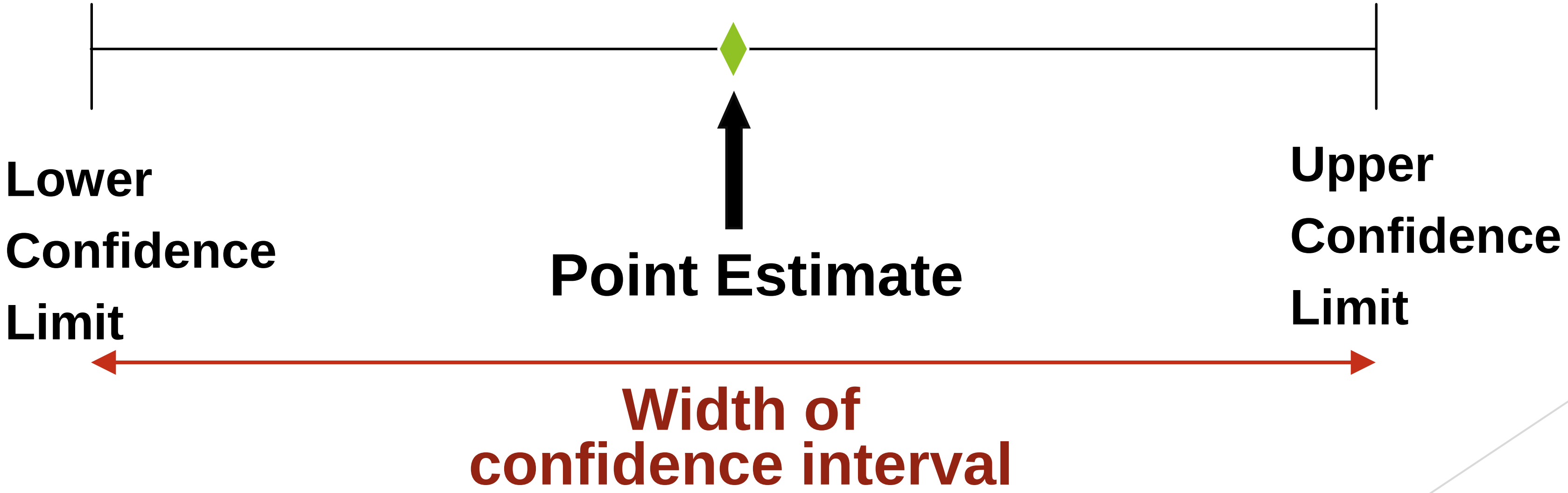
Chapter Outline

Content of this chapter

- Confidence Intervals for the Population Mean, μ
  - when Population Standard Deviation σ is Known
  - when Population Standard Deviation σ is Unknown
- Determining the Required Sample Size

Point and Interval Estimates

- A point estimate is a single number,
- a confidence interval provides additional information about the variability of the estimate



Point Estimates

We can estimate a Population Parameter …		with a Sample Statistic (a Point Estimate)
Mean	μ	$\bar{X}$
Proportion	π	p

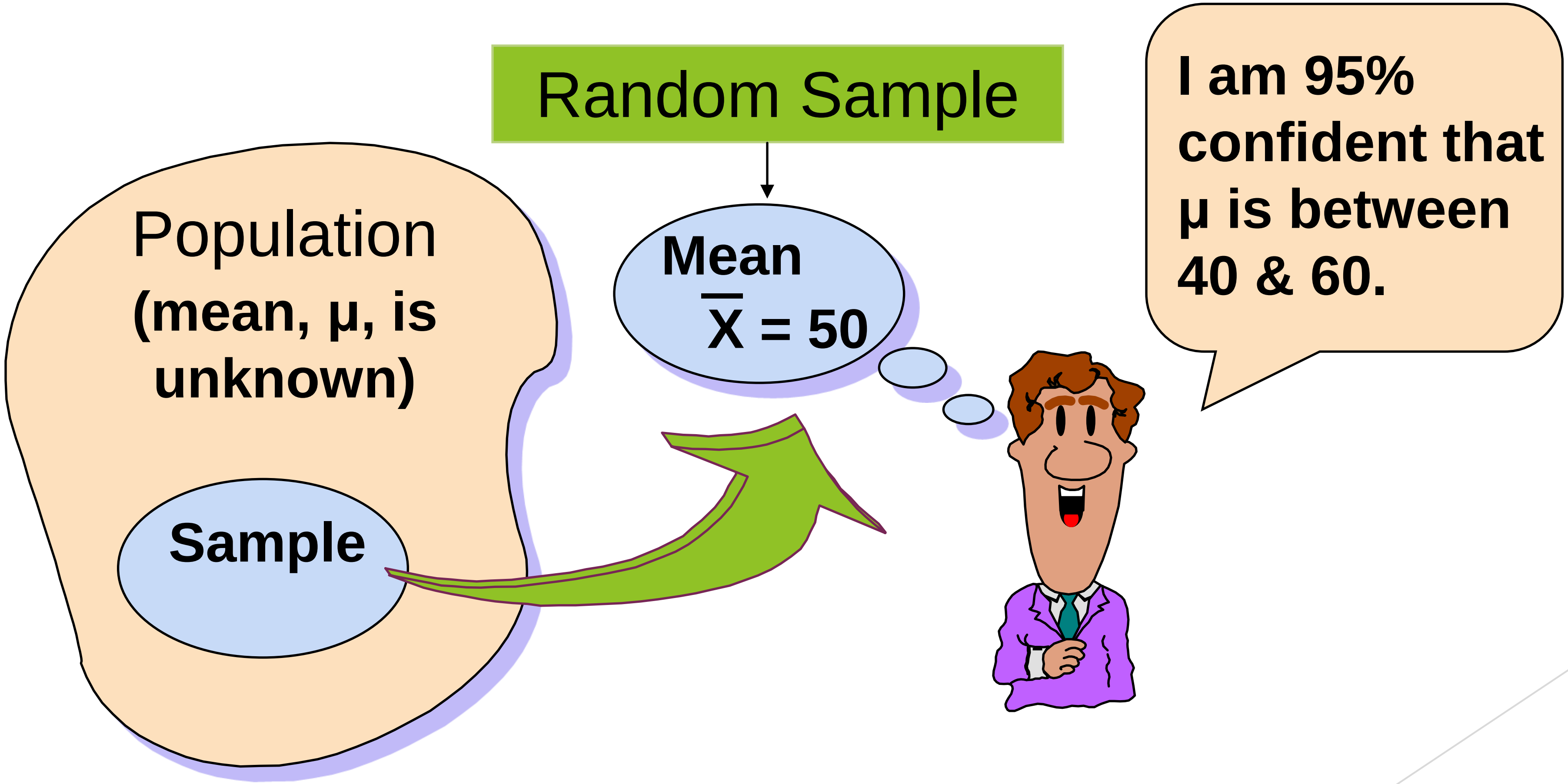
Confidence Intervals

- How much uncertainty is associated with a point estimate of a population parameter?
- An interval estimate provides more information about a population characteristic than does a point estimate
- Such interval estimates are called confidence intervals

Confidence Interval Estimate

- An interval gives a range of values:
  - Takes into consideration variation in sample statistics from sample to sample
  - Based on observations from 1 sample
  - Gives information about closeness to unknown population parameters
  - Stated in terms of level of confidence
    - e.g. 95% confident, 99% confident
    - Can never be 100% confident

Estimation Process



General Formula

- The general formula for all confidence intervals is:

Point Estimate ± (Critical Value)(Standard Error)

Where:

- Point Estimate is the sample statistic estimating the population parameter of interest

- Critical Value is a table value based on the sampling distribution of the point estimate and the desired confidence level

- Standard Error is the standard deviation of the point estimate

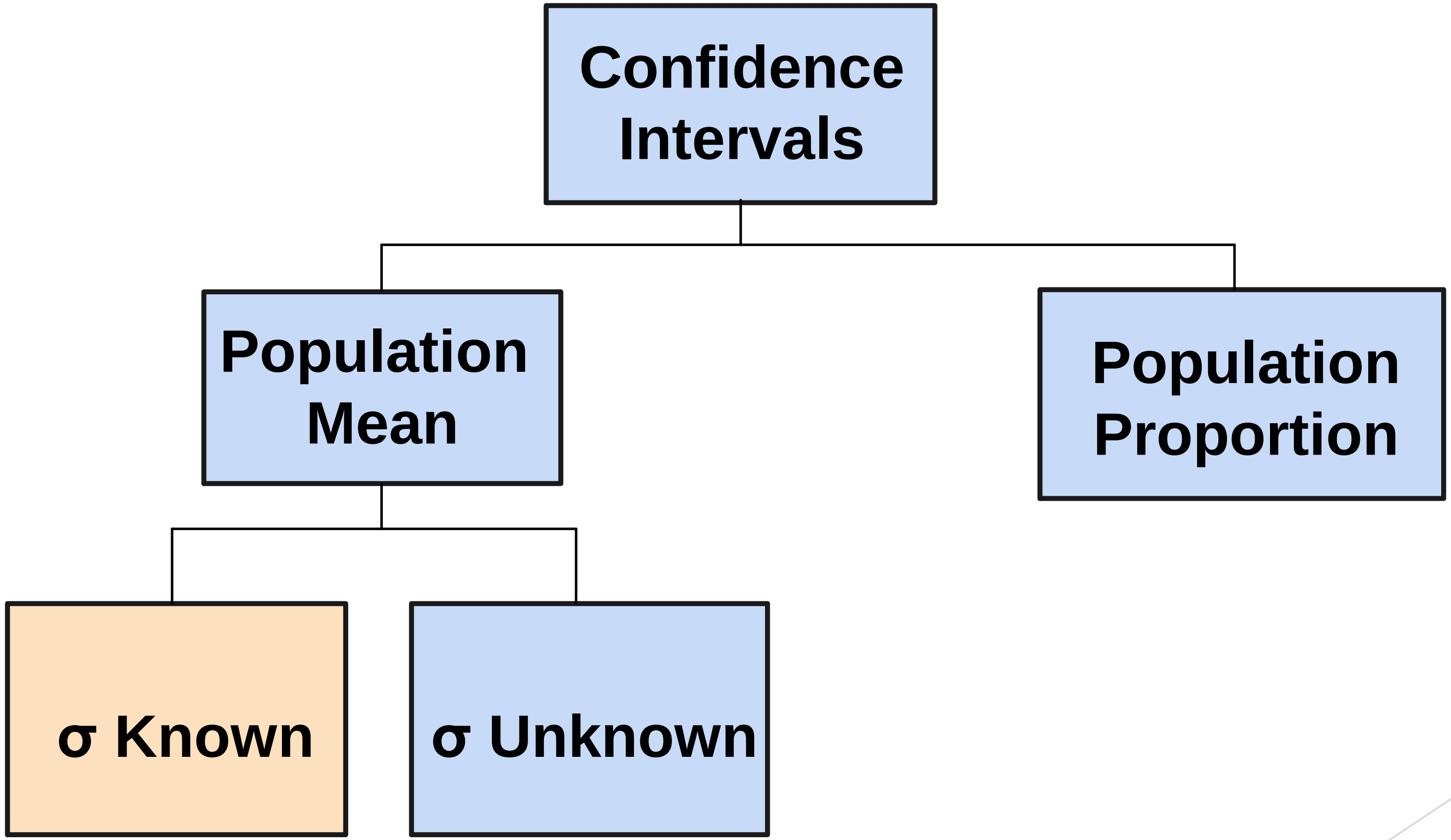
Confidence Level

- Confidence the interval will contain the unknown population parameter
- A percentage (less than 100%)

Confidence Level, (1-α)

- Suppose confidence level = 95%
- Also written (1 - α) = 0.95, (so α = 0.05)
- A relative frequency interpretation:
  - 95% of all the confidence intervals that can be constructed will contain the unknown true parameter
- A specific interval either will contain or will not contain the true parameter
  - No probability involved in a specific interval

Confidence Intervals



Confidence Interval for μ (σ Known)

- Assumptions
  - Population standard deviation σ is known
  - Population is normally distributed
  - If population is not normal, use large sample (n > 30)
- Confidence interval estimate:

$$\bar{X} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

where

$\bar{X}$ is the point estimate

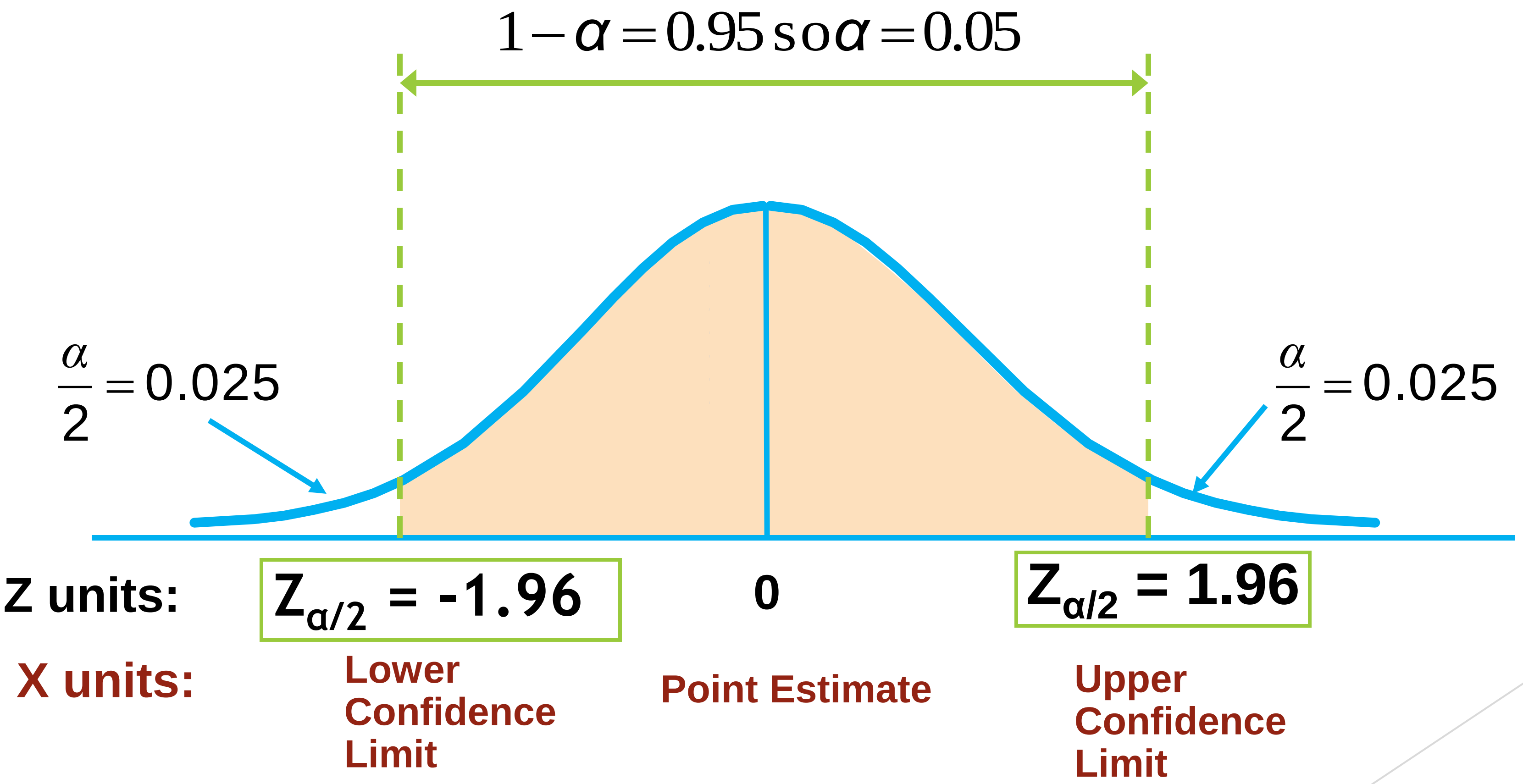
$Z_{\alpha/2}$ is the normal distribution critical value for a probability of $\alpha/2$ in each tail

$\sigma/\sqrt{n}$ is the standard error

Finding the Critical Value, $Z_{\alpha/2}$

- Consider a 95% confidence interval:

$$Z_{\alpha/2} = \pm 1.96$$



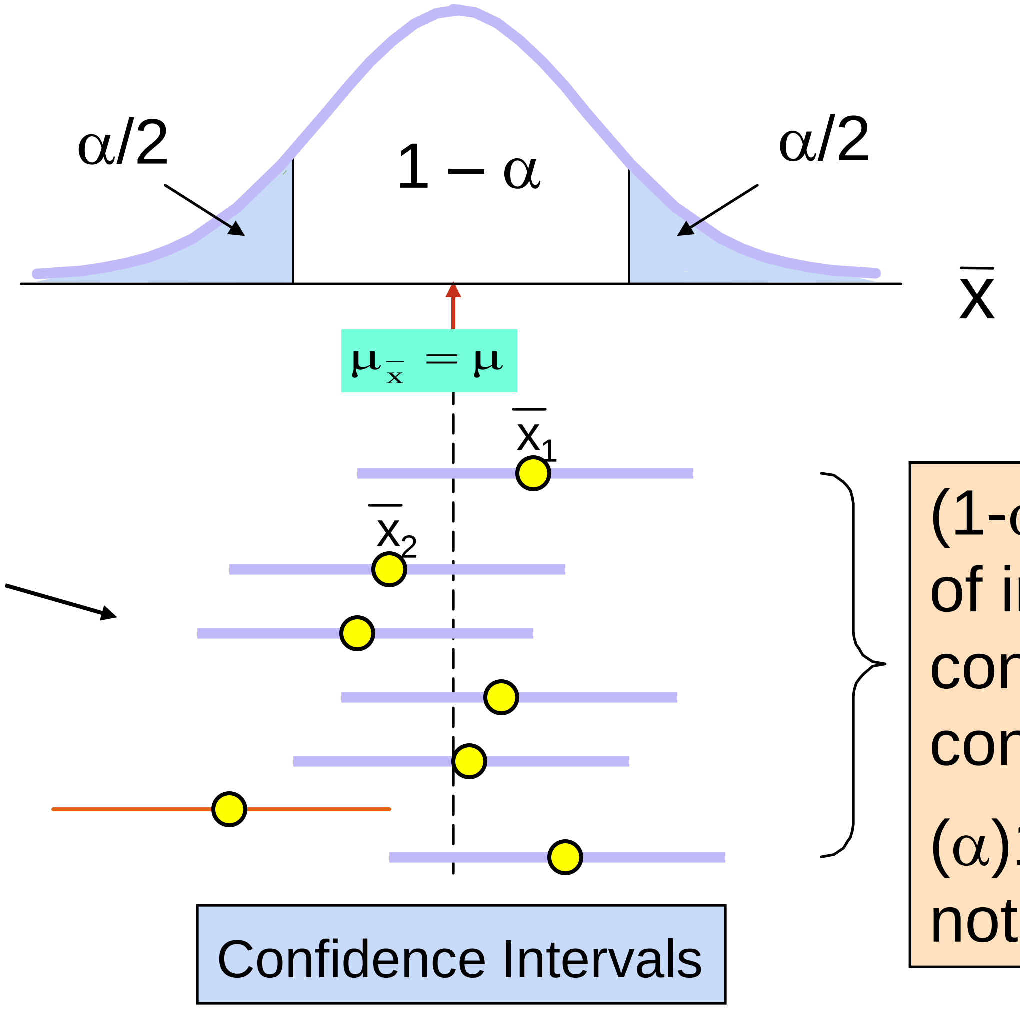
Common Levels of Confidence

- Commonly used confidence levels are 90%, 95%, and 99%

Confidence Level	***Confidence Coefficient,*** $1-\alpha$	$Z_{\alpha/2}$ ***value***
80%	**0.80**	**1.28**
90%	**0.90**	**1.645**
95%	**0.95**	**1.96**
98%	**0.98**	**2.33**
99%	**0.99**	**2.58**
99.8%	**0.998**	**3.08**
99.9%	**0.999**	**3.27**

Intervals and Level of Confidence

Sampling Distribution of the Mean



Intervals extend from

$$\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

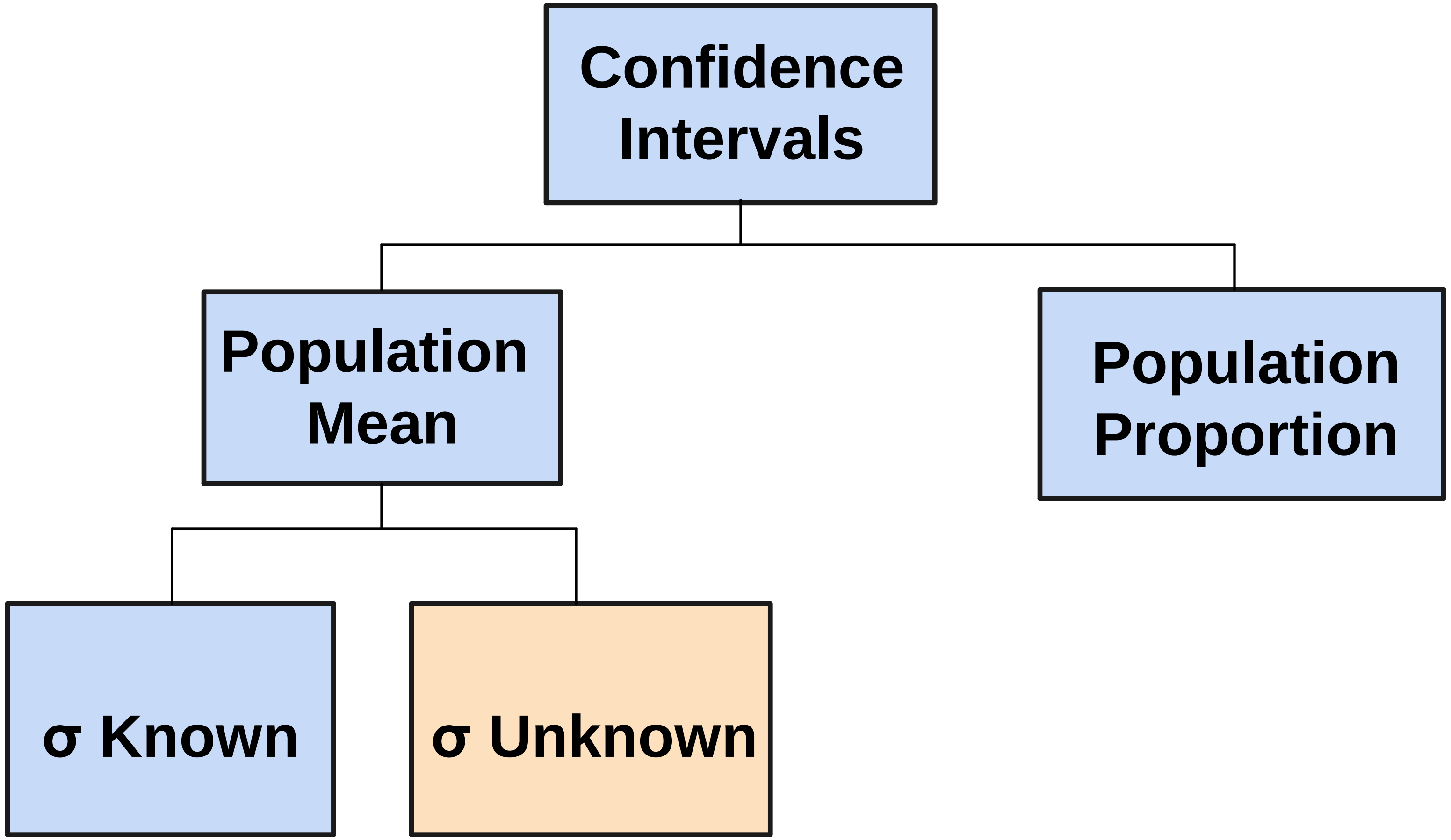
to

$$\bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

(1-α)100% of intervals constructed contain μ;

(α)100% do not.

Confidence Intervals



Do You Ever Truly Know σ?

- Probably not!
- In virtually all real world business situations, σ is not known.
- If there is a situation where σ is known then μ is also known (since to calculate σ you need to know μ.)
- If you truly know μ there would be no need to gather a sample to estimate it.

Confidence Interval for μ (σ Unknown)

- If the population standard deviation σ is unknown, we can substitute the sample standard deviation, S
- This introduces extra uncertainty, since S is variable from sample to sample
- So we use the t distribution instead of the normal distribution

Confidence Interval for μ (σ Unknown)

- Assumptions
  - Population standard deviation is unknown
  - Population is normally distributed
  - If population is not normal, use large sample (n > 30)
- Use Student's t Distribution
- Confidence Interval Estimate:

$$\overline{X} \pm t_{\alpha/2} \frac{S}{\sqrt{n}}$$

(where $t_{\alpha/2}$ is the critical value of the t distribution with n -1 degrees of freedom and an area of α/2 in each tail)

Degrees of Freedom (df)

Idea: Number of observations that are free to vary after sample mean has been calculated

Example: Suppose the mean of 3 numbers is 8.0

Let $X_1 = 7$

Let $X_2 = 8$

What is X_3?

→

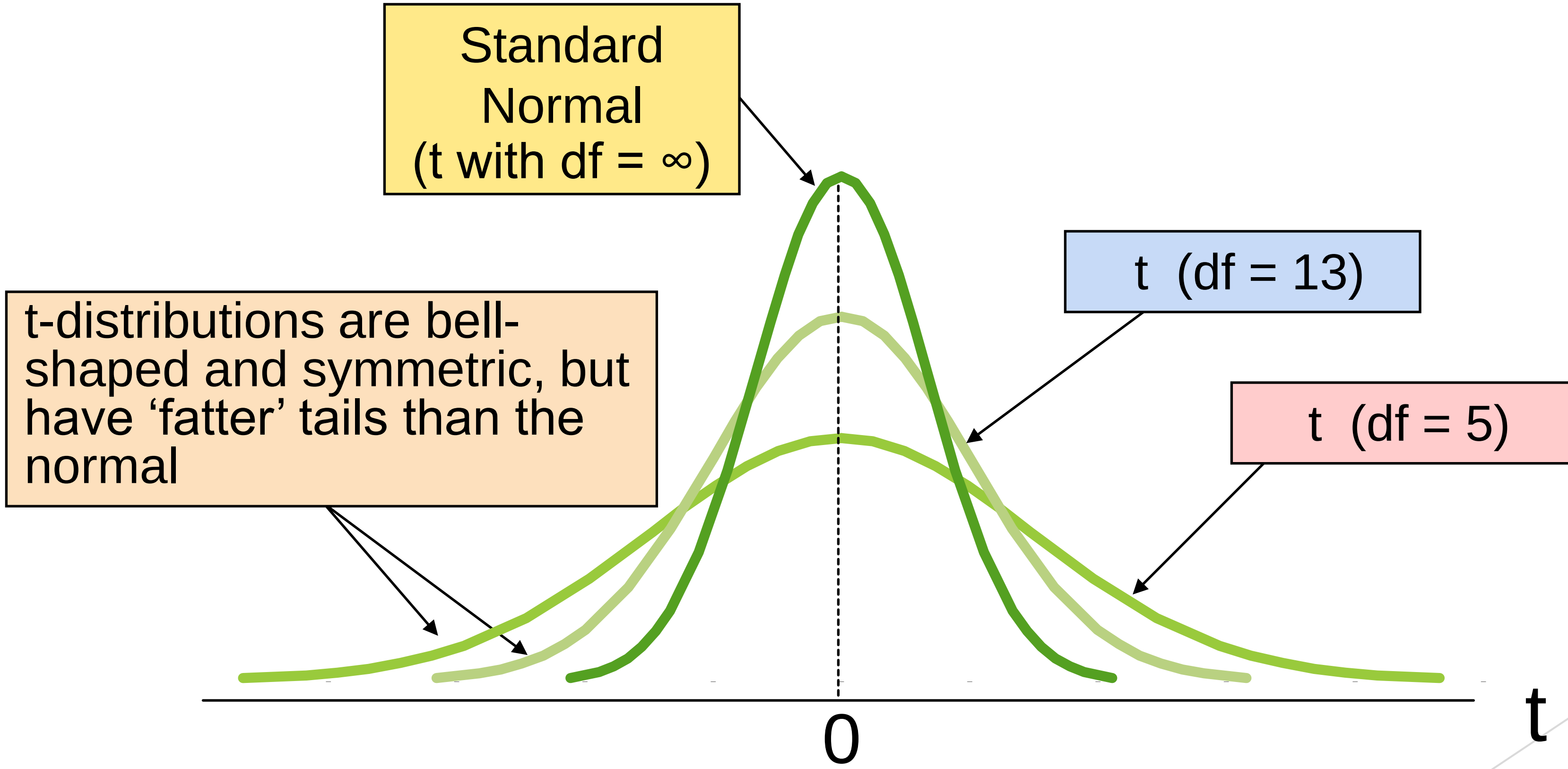
If the mean of these three values is 8.0,
then X_3 must be 9
(i.e., X_3 is not free to vary)

Here, n = 3, so degrees of freedom $= n – 1 = 3 – 1 = 2$

(2 values can be any numbers, but the third is not free to vary for a given mean)

Student's t Distribution

Note: $t \longrightarrow Z$ as n increases



Example of t distribution confidence interval

A random sample of n = 25 has $\bar{X}$ = 50 and S = 8. Form a 95% confidence interval for μ

- d.f. = n – 1 = 24, so $t_{\alpha/2} = t_{0.025} = 2.0639$

The confidence interval is

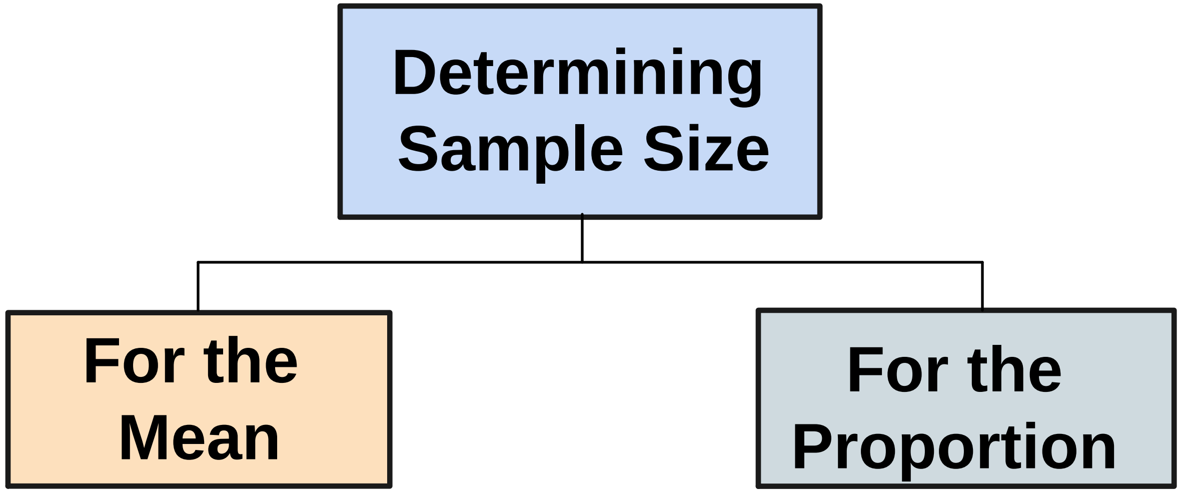
$$\bar{X} \pm t_{\alpha/2} \frac{S}{\sqrt{n}} = 50 \pm (2.0639) \frac{8}{\sqrt{25}}$$

$$46.698 \le \mu \le 53.302$$

Example of t distribution confidence interval

- Interpreting this interval requires the assumption that the population you are sampling from is approximately a normal distribution (especially since n is only 25).
- This condition can be checked by creating a:
  - Normal probability plot or
  - Boxplot

Determining Sample Size



Sampling Error

- The required sample size can be found to reach a desired margin of error (e) with a specified level of confidence ($1 - \alpha$)

- The margin of error is also called sampling error
  - the amount of imprecision in the estimate of the population parameter
  - the amount added and subtracted to the point estimate to form the confidence interval

Determining Sample Size

Determining Sample Size

For the Mean

$$\bar{X} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$e = Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Determining Sample Size

Determining Sample Size

For the Mean

$$e = Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Now solve for n to get

$$n = \frac{Z_{\alpha/2}^{2}\sigma^{2}}{e^{2}}$$

Determining Sample Size

- To determine the required sample size for the mean, you must know:
  - The desired level of confidence $(1 - \alpha)$, which determines the critical value, $Z_{\alpha/2}$
  - The acceptable sampling error, e
  - The standard deviation, σ

Required Sample Size Example

If σ = 45, what sample size is needed to estimate the mean within ± 5 with 90% confidence?

$$n = \frac{Z^2 \sigma^2}{e^2} = \frac{(1.645)^2 (45)^2}{5^2} = 219.19$$

So the required sample size is **n = 220**

(Always round up)

If σ is unknown

- If unknown, σ can be estimated when using the required sample size formula
  - Use a value for σ that is expected to be at least as large as the true σ
  - Select a pilot sample and estimate σ with the sample standard deviation, S

hypothesis testing

Objectives

In this session, you learn:

- The basic principles of hypothesis testing
- How to use hypothesis testing to test a mean or proportion
- The assumptions of each hypothesis-testing procedure, how to evaluate them, and the consequences if they are seriously violated
- Pitfalls & ethical issues involved in hypothesis testing
- How to avoid the pitfalls involved in hypothesis testing

What is a Hypothesis?

- A hypothesis is a claim (assertion) about a population parameter:
  - population mean

Example: The mean monthly cell phone bill in this city is $\mu = \$42$

  - population proportion

Example: The proportion of adults in this city with cell phones is $\pi = 0.68$

The Null Hypothesis, H_0

- States the claim or assertion to be tested

Example: The mean diameter of a manufactured bolt is 30mm ($H_0 : \mu = 30$)

- Is always about a population parameter, not about a sample statistic

$$H_0 : \mu = 30$$

$$\cancel{H_0 : \bar{X} = 30}$$

The Null Hypothesis, H_0

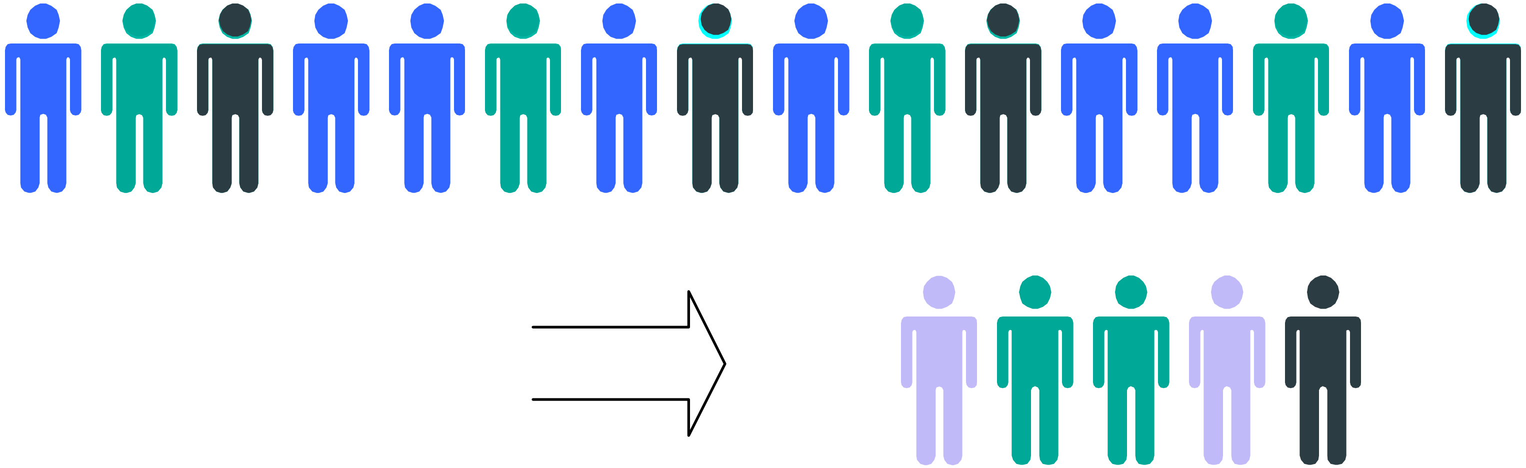
- Begin with the assumption that the null hypothesis is true
- Always contains “=“, or “≤”, or “≥” sign
- May or may not be rejected

The Alternative Hypothesis, H_1

- Is the opposite of the null hypothesis
  - e.g., The mean diameter of a manufactured bolt is not equal to 30mm (H_1: $\mu \neq 30$)
- Never contains the "=", or "≤", or "≥" sign
- May or may not be proven
- Is generally the hypothesis that the researcher is trying to prove

The Hypothesis Testing Process

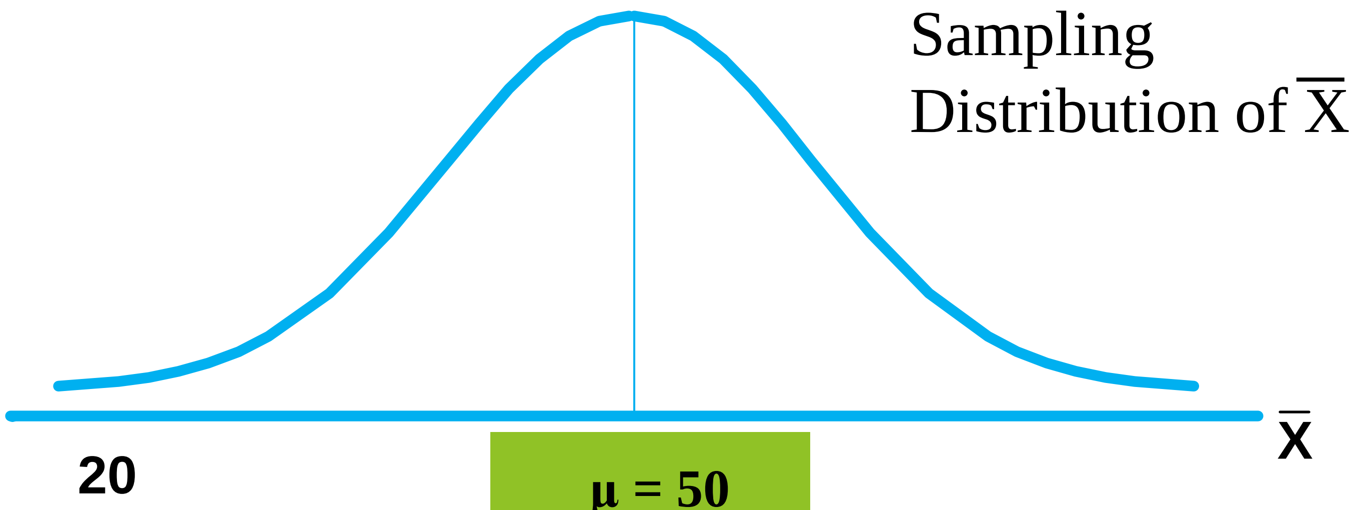
- Claim: The population mean age is 50.
  - H_0: $\mu = 50$, H_1: $\mu \neq 50$
- Sample the population and find the sample mean.



The Hypothesis Testing Process

- Suppose the sample mean age was $X = 20$.
- This is significantly lower than the claimed mean population age of 50.
- If the null hypothesis were true, the probability of getting such a different sample mean would be very small, so you reject the null hypothesis.
- In other words, getting a sample mean of 20 is so unlikely if the population mean was 50, you conclude that the population mean must not be 50.

The Hypothesis Testing Process



$\mu = 50$
If H_0 is true

If it is unlikely that you would get a sample mean of this value ...

... When in fact this were the population mean…

... then you reject the null hypothesis that $\mu = 50$.

The Test Statistic and Critical Values

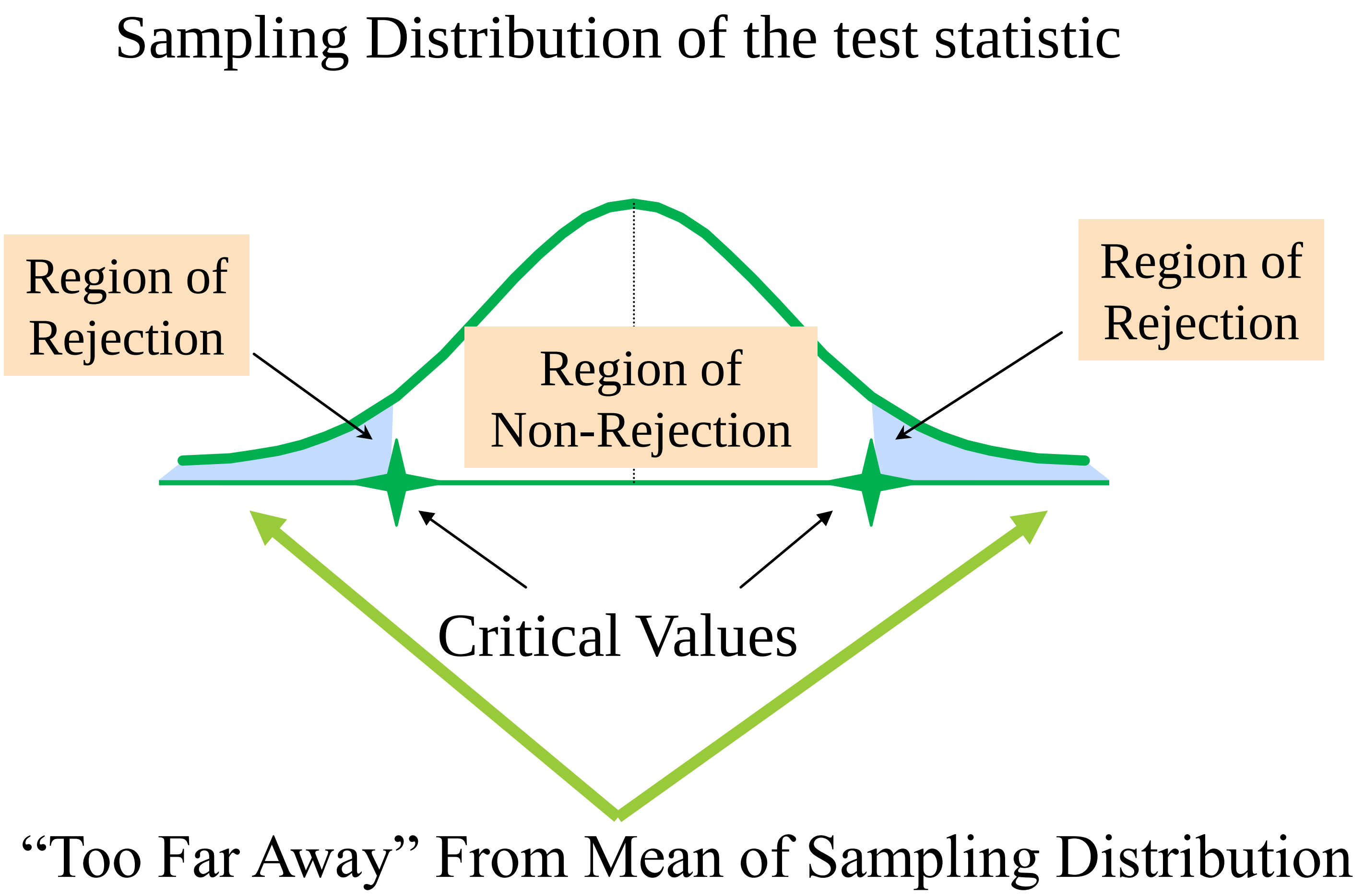
- If the sample mean is close to the stated population mean, the null hypothesis is not rejected.

- If the sample mean is far from the stated population mean, the null hypothesis is rejected.

- How far is "far enough" to reject H_0?

- The critical value of a test statistic creates a "line in the sand" for decision making -- it answers the question of how far is far enough.

The Test Statistic and Critical Values



Risks in Decision Making Using Hypothesis Testing

- **Type I Error**
  - Reject a true null hypothesis
  - A type I error is a “false alarm”
  - The probability of a Type I Error is α
    - Called level of significance of the test
    - Set by researcher in advance
- **Type II Error**
  - Failure to reject a false null hypothesis
  - Type II error represents a “missed opportunity”
  - The probability of a Type II Error is β

Possible Errors in Hypothesis Test Decision Making

Possible Hypothesis Test Outcomes		
	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	No Error Probability $1 - \alpha$	Type II Error Probability β
Reject H_0	Type I Error Probability α	No Error Power $1 - \beta$

Possible Errors in Hypothesis Test Decision Making

- The confidence coefficient (1-α) is the probability of not rejecting H_0 when it is true.

- The confidence level of a hypothesis test is (1-α)*100%.

- The power of a statistical test (1-β) is the probability of rejecting H_0 when it is false.

Type I & II Error Relationship

- Type I and Type II errors cannot happen at the same time
  - A Type I error can only occur if H_0 is true
  - A Type II error can only occur if H_0 is false

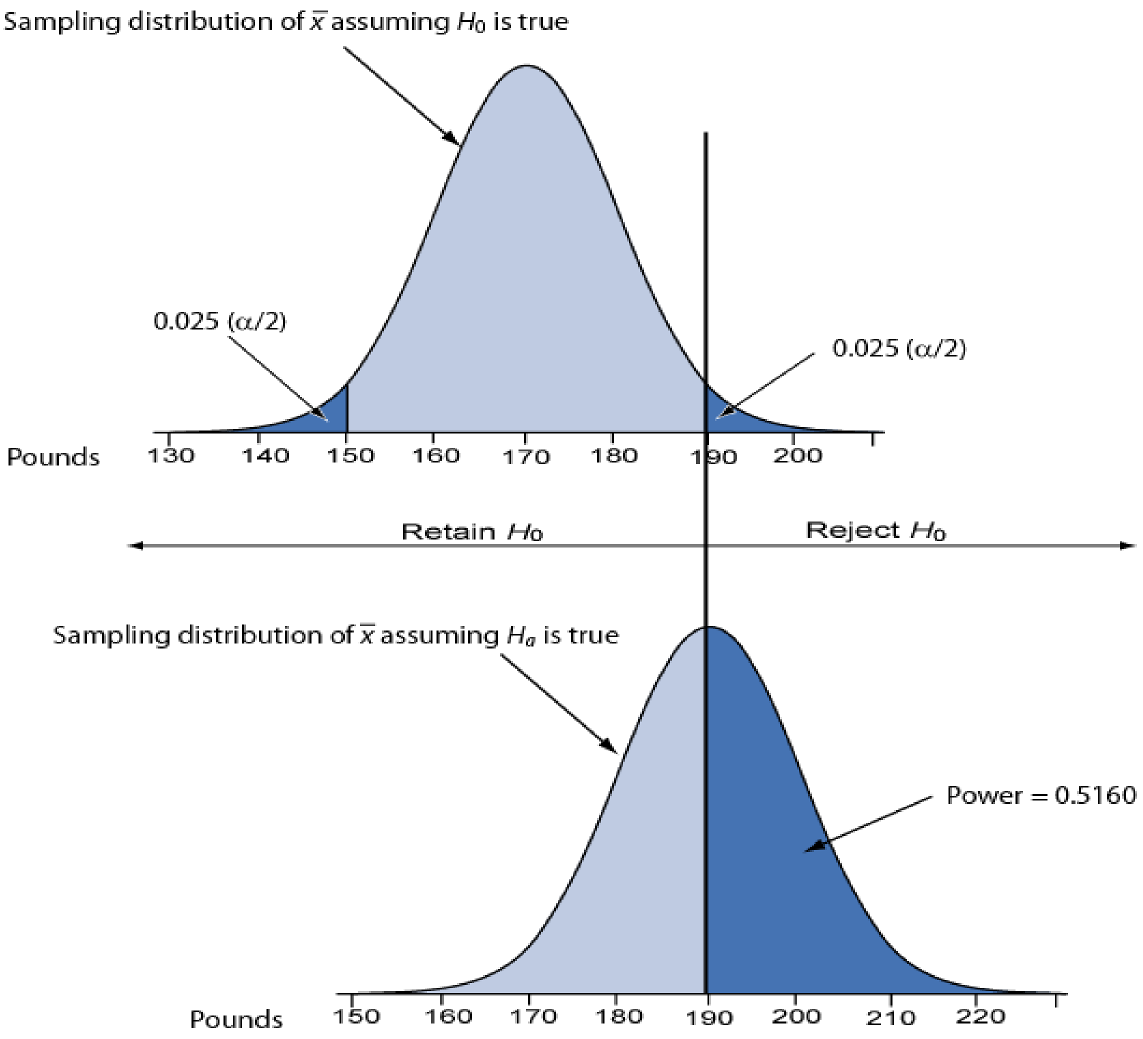
If Type I error probability (α) ↑, then

Type II error probability (β) ↓

Factors Affecting Type II Error

- All else equal,
  - β ↑ when the difference between hypothesized parameter and its true value ↓
  - β ↑ when α ↓
  - β ↑ when σ ↑
  - β ↑ when *n* ↓

Power

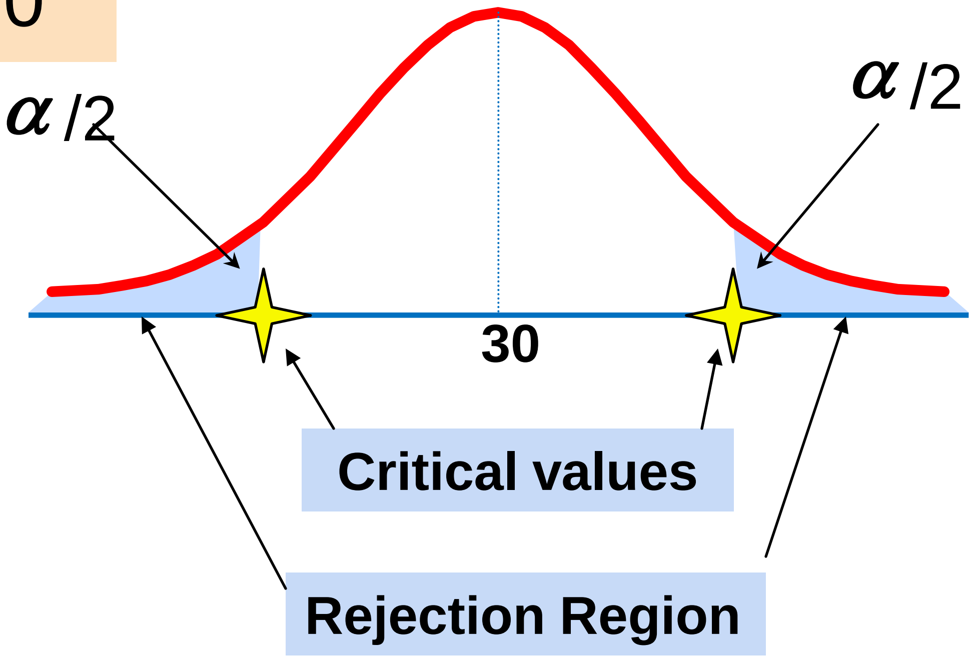


Sampling distribution of $\bar{x}$ assuming H_0 is true
0.025 ($\alpha/2$)
0.025 ($\alpha/2$)
Pounds
130 140 150 160 170 180 190 200
Retain H_0
Reject H_0
Sampling distribution of $\bar{x}$ assuming H_a is true
Power = 0.5160
Pounds
150 160 170 180 190 200 210 220

Level of Significance and the Rejection Region

H_0: μ = 30
H_1: μ ≠ 30

Level of significance = α



This is a two-tail test because there is a rejection region in both tails

Hypothesis Tests for the Mean

- **Hypothesis Tests for μ**
  - **σ Known (Z test)**
  - **σ Unknown (t test)**

Z Test of Hypothesis for the Mean (σ Known)

- Convert sample statistic ($\bar{X}$) to a Z_{STAT} test statistic

Hypothesis Tests for μ

σ Known (Z test)

σ Unknown (t test)

The test statistic is:

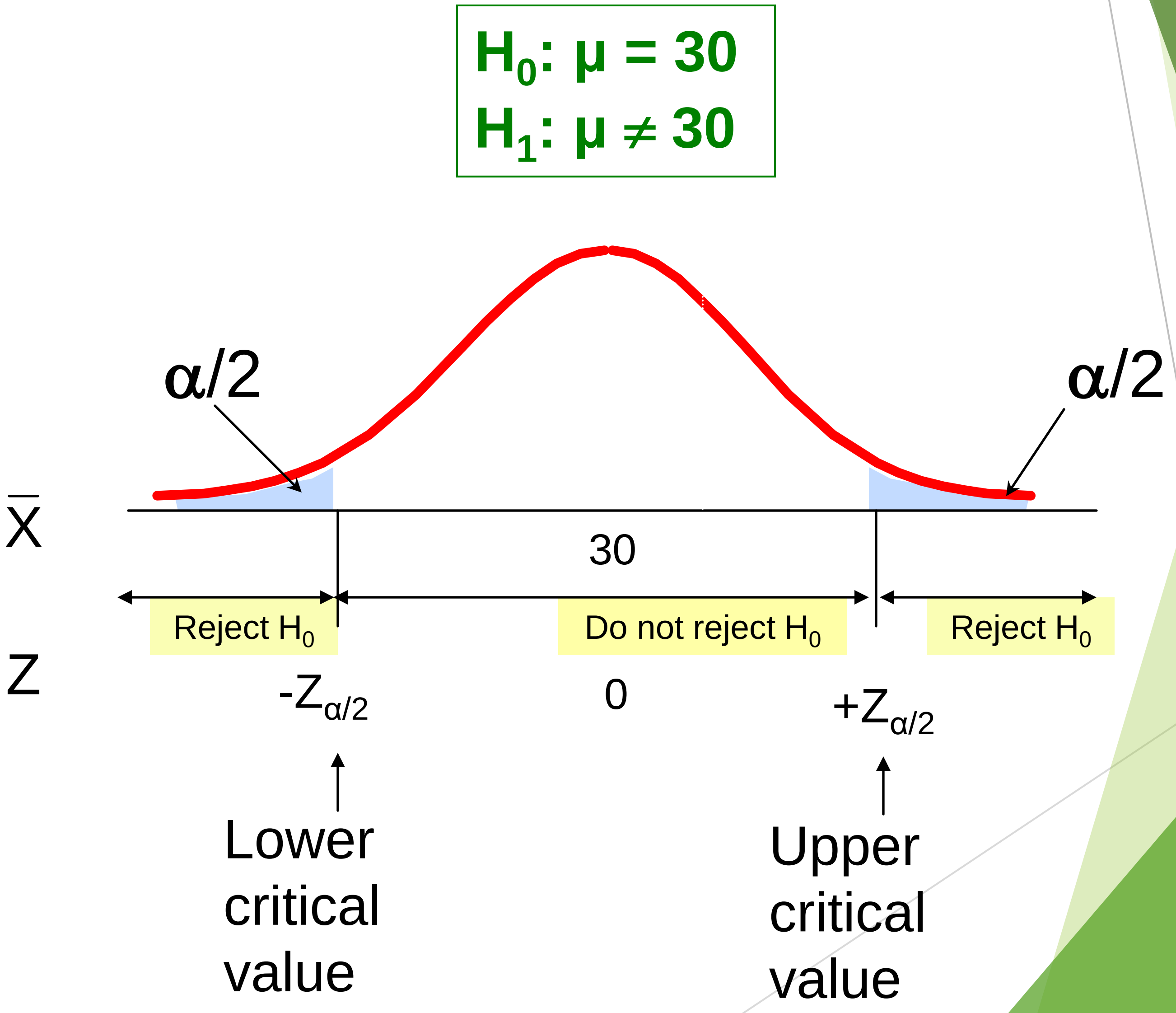
$$Z_{STAT} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Critical Value Approach to Testing

- For a two-tail test for the mean, σ known:
- Convert sample statistic ($\bar{X}$) to test statistic (Z_{STAT})
- Determine the critical Z values for a specified level of significance α from a table or by using computer software
- Decision Rule: If the test statistic falls in the rejection region, reject H_0 ; otherwise do not reject H_0

Two-Tail Tests

- There are two cutoff values (critical values), defining the regions of rejection



6 Steps in Hypothesis Testing

1. State the null hypothesis, H_0 and the alternative hypothesis, $H_{1.}$
2. Choose the level of significance, α, and the sample size, n. The level of significance is based on the relative importance of Type I and Type II errors.
3. Determine the appropriate test statistic and sampling distribution.
4. Determine the critical values that divide the rejection and nonrejection regions.

6 Steps in Hypothesis Testing

5. Collect data and compute the value of the test statistic.
6. Make the statistical decision and state the managerial conclusion. If the test statistic falls into the nonrejection region, do not reject the null hypothesis H_0. If the test statistic falls into the rejection region, reject the null hypothesis. Express the managerial conclusion in the context of the problem.

Hypothesis Testing Example

Test the claim that the true mean diameter of a manufactured bolt is 30mm. (Assume $\sigma = 0.8$)

1. State the appropriate null and alternative hypotheses
   - H_0: $\mu = 30$ H_1: $\mu \neq 30$ (This is a two-tail test)
2. Specify the desired level of significance and the sample size
   - Suppose that $\alpha = 0.05$ and $n = 100$ are chosen for this test

Hypothesis Testing Example

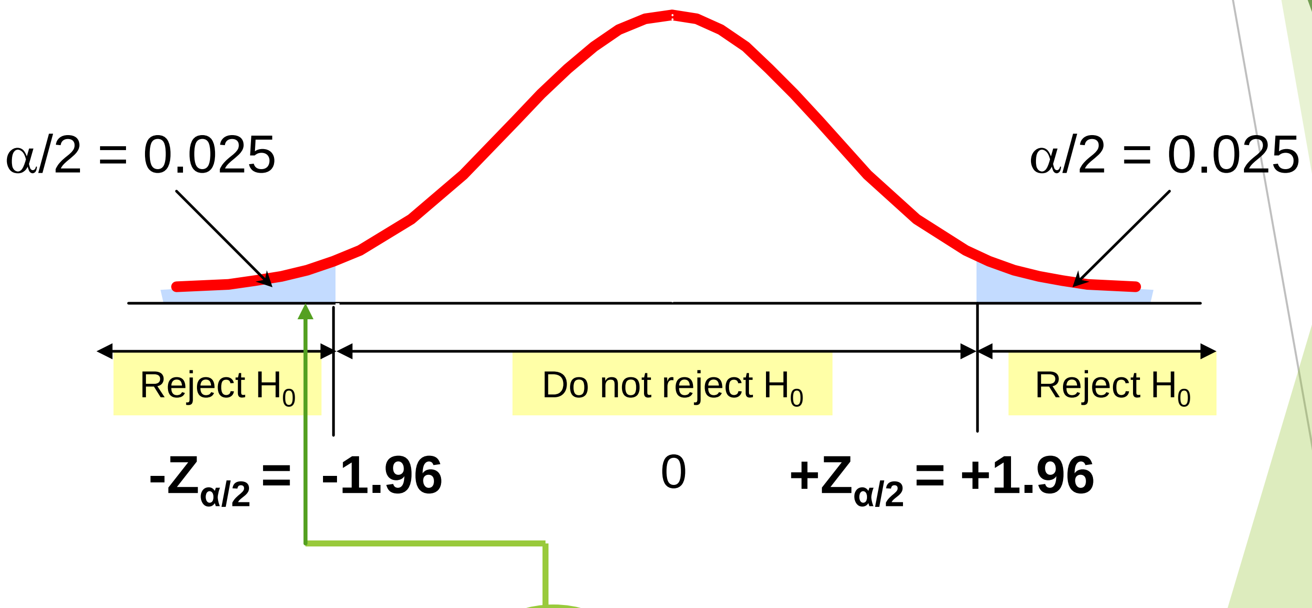
3. Determine the appropriate technique
   - σ is assumed known so this is a Z test
4. Determine the critical values
   - For $\alpha = 0.05$ the critical Z values are ±1.96
5. Collect the data and compute the test statistic
   - Suppose the sample results are $n = 100$, $\bar{X} = 29.84$ ($\sigma = 0.8$ is assumed known)

So the test statistic is:

$$Z_{STAT} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{29.84 - 30}{\frac{0.8}{\sqrt{100}}} = \frac{-0.16}{0.08} = -2.0$$

Hypothesis Testing Example

6. Is the test statistic in the rejection region?



Reject H_0 if $Z_{STAT} < -1.96$ or $Z_{STAT} > 1.96$; otherwise do not reject H_0

Here, $Z_{STAT} = -2.0 < -1.96$, so the test statistic is in the rejection region

p-Value Approach to Testing

- p-value: Probability of obtaining a test statistic equal to or more extreme than the observed sample value given H_0 is true.
  - The p-value is also called the observed level of significance.
  - It is the smallest value of α for which H_0 can be rejected .

p-Value Approach to Testing: Interpreting the p-value

- Compare the p-value with α
  - If p-value $< \alpha$, reject H_0
  - If p-value $\geq \alpha$, do not reject H_0
- Remember
  - If the p-value is low then H_0 must go

The 5 Step p-value approach to Hypothesis Testing

1. State the null hypothesis, H_0 and the alternative hypothesis, $H_{1.}$

2. Choose the level of significance, α, and the sample size, n. The level of significance is based on the relative importance of the risks of a type I and a type II error.

3. Determine the appropriate test statistic and sampling distribution.

4. Collect data and compute the value of the test statistic and the p-value.

5. Make the statistical decision and state the managerial conclusion. If the p-value is < α then reject H_0, otherwise do not reject H_0. State the managerial conclusion in the context of the problem.

p-value Hypothesis Testing Example

Test the claim that the true mean diameter of a manufactured bolt is 30mm.
(Assume $\sigma = 0.8$)

1. State the appropriate null and alternative hypotheses.
   - H_0: $\mu = 30$ H_1: $\mu \neq 30$ (This is a two-tail test)
2. Specify the desired level of significance and the sample size
   - Suppose that $\alpha = 0.05$ and $n = 100$ are chosen for this test

p-value Hypothesis Testing Example

3. Determine the appropriate technique
   - σ is assumed known so this is a Z test.
4. Collect the data, compute the test statistic and the p-value
   - Suppose the sample results are

     n = 100, $\bar{X}$ = 29.84 (σ = 0.8 is assumed known)

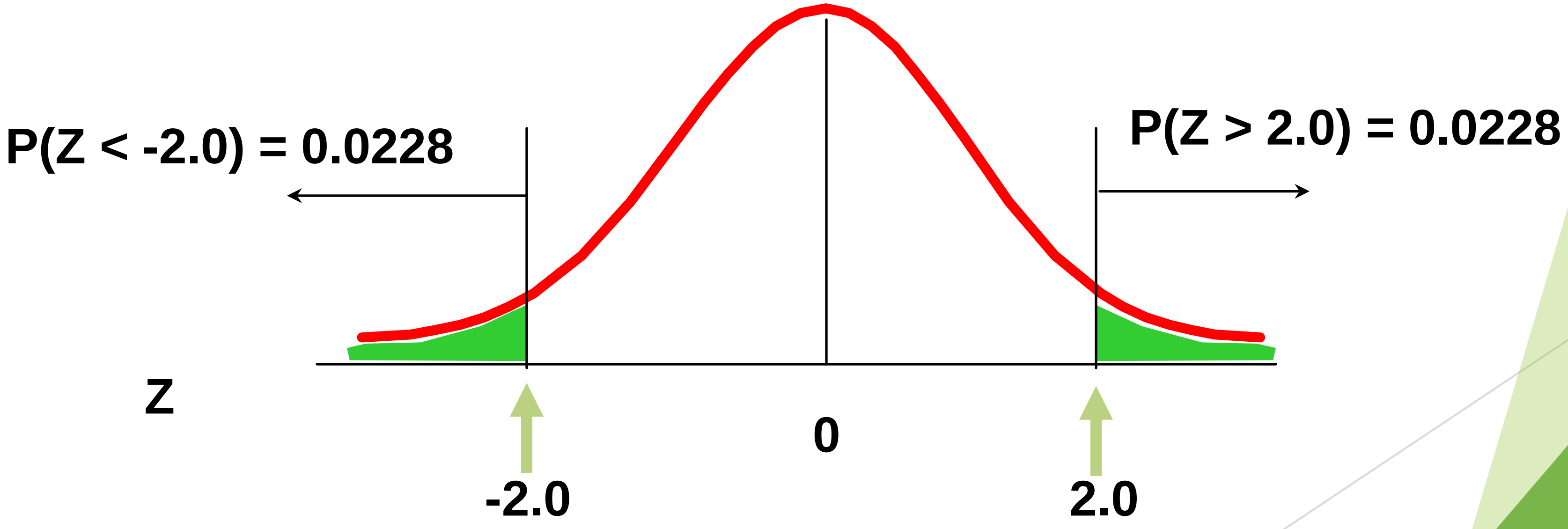
   So the test statistic is:

$$Z_{STAT} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{29.84 - 30}{\frac{0.8}{\sqrt{100}}} = \frac{-0.16}{0.08} = -2.0$$

p-Value Hypothesis Testing Example: Calculating the p-value

4. (continued) Calculate the p-value.

- How likely is it to get a Z_{STAT} of -2 (or something further from the mean (0), in either direction) if H_0 is true?



p-value = 0.0228 + 0.0228 = 0.0456

p-value Hypothesis Testing Example

- 5. Is the p-value < α?
  - Since p-value = 0.0456 < α = 0.05 Reject H_0
- 5. (continued) State the managerial conclusion in the context of the situation.
  - There is sufficient evidence to conclude the mean diameter of a manufactured bolt is not equal to 30mm.

Connection Between Two Tail Tests and Confidence Intervals

- For X = 29.84, σ = 0.8 and n = 100, the 95% confidence interval is:

$$29.84 - (1.96)\frac{0.8}{\sqrt{100}} \quad \text{to} \quad 29.84 + (1.96)\frac{0.8}{\sqrt{100}}$$

$$29.6832 \le \mu \le 29.9968$$

- Since this interval does not contain the hypothesized mean (30), we reject the null hypothesis at $\alpha = 0.05$

Do You Ever Truly Know σ?

- Probably not!
- In virtually all real world business situations, σ is not known.
- If there is a situation where σ is known then μ is also known (since to calculate σ you need to know μ.)
- If you truly know μ there would be no need to gather a sample to estimate it.

Hypothesis Testing: σ Unknown

- If the population standard deviation is unknown, you instead use the sample standard deviation S.
- Because of this change, you use the t distribution instead of the Z distribution to test the null hypothesis about the mean.
- When using the t distribution you must assume the population you are sampling from follows a normal distribution.
- All other steps, concepts, and conclusions are the same.

t Test of Hypothesis for the Mean (σ Unknown)

- Convert sample statistic ($\bar{X}$) to a t_{STAT} test statistic

Hypothesis Tests for μ

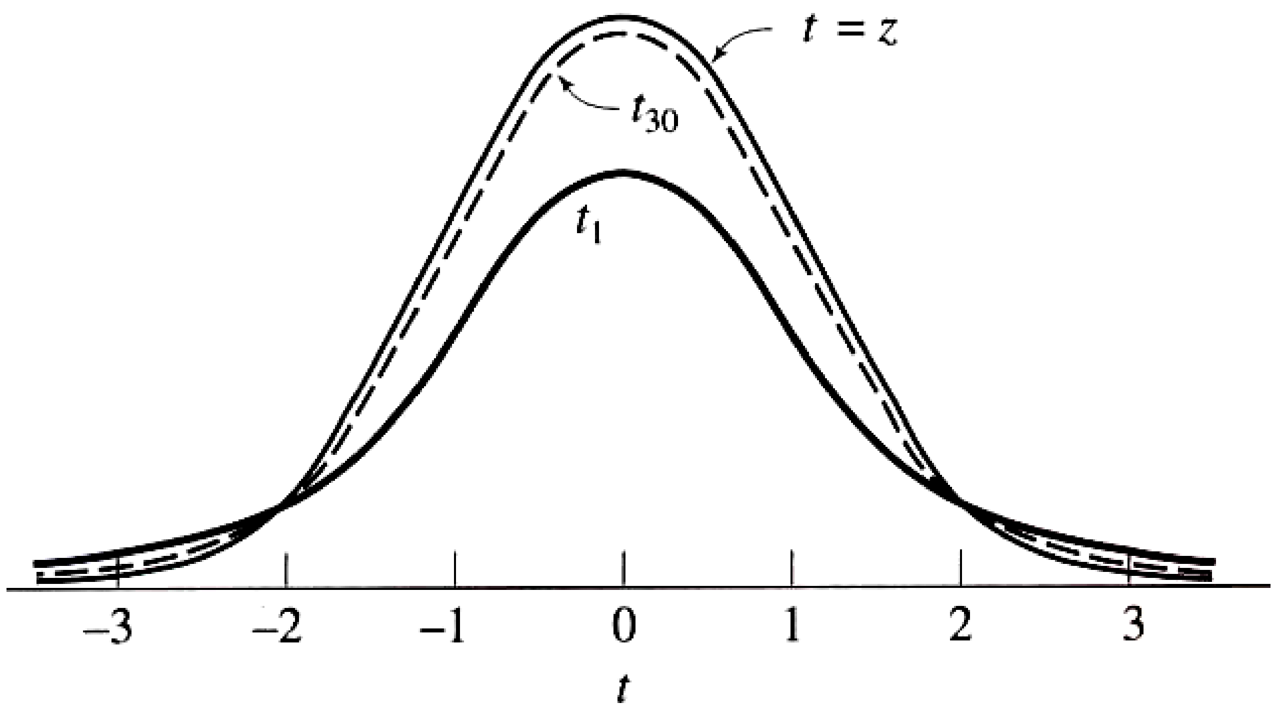
σ Known (Z test)

σ Unknown (t test)

The test statistic is:

$$t_{STAT} = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$$

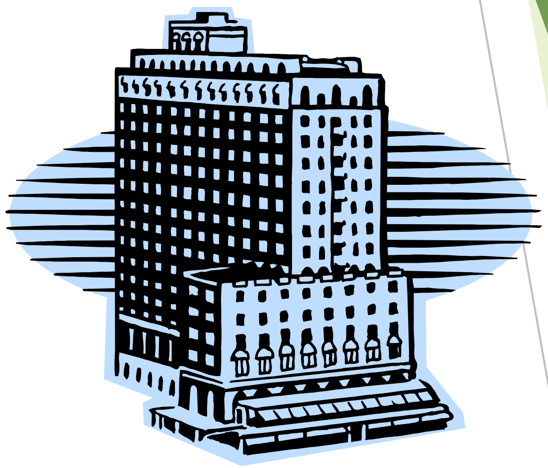
Distribution of Z & t



Example: Two-Tail Test (σ Unknown)

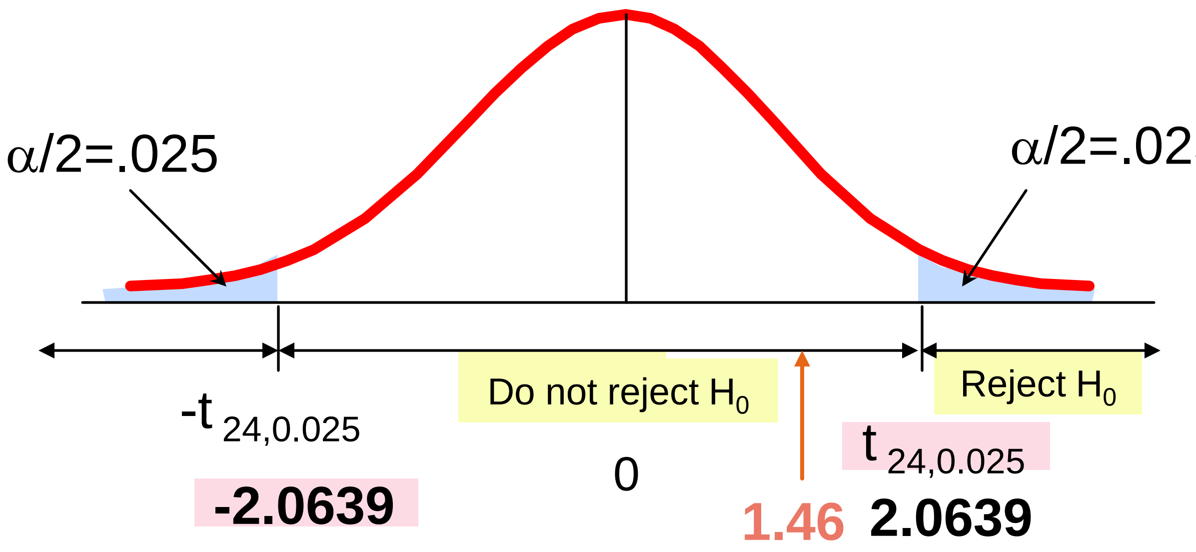
The average cost of a hotel room in New York is said to be \$168 per night. To determine if this is true, a random sample of 25 hotels is taken and resulted in an $\bar{X}$ of \$172.50 and an S of \$15.40. Test the appropriate hypotheses at α = 0.05.

(Assume the population distribution is normal)



Example Solution: Two-Tail t Test

H_0: $\mu = 168$
H_1: $\mu \neq 168$



- $\alpha = 0.05$
- $n = 25$, $df = 25-1=24$
- σ is unknown, so
- use a t statistic
- Critical Value:
- $\pm t_{24,0.025} = \pm\ 2.0639$

$$t_{STAT} = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = 1.46$$

Do not reject H_0: insufficient evidence that true mean cost is different from $168

To Use the t-test Must Assume the Population Is Normal

- As long as the sample size is not very small and the population is not very skewed, the t-test can be used.

- To evaluate the normality assumption:
  - Determine how closely sample statistics match the normal distribution's theoretical properties.
  - Construct a histogram or stem-and-leaf display or boxplot or a normal probability plot.

Connection of Two Tail Tests to Confidence Intervals

- For $\bar{X} = 172.5$, $S = 15.40$ and $n = 25$, the 95% confidence interval for μ is:

$$172.5 - (2.0639)\ 15.4/\sqrt{25} \quad \text{to} \quad 172.5 + (2.0639)\ 15.4/\sqrt{25}$$

$$166.14 \leq \mu \leq 178.86$$

- Since this interval contains the Hypothesized mean (168), we do not reject the null hypothesis at $\alpha = 0.05$

One-Tail Tests

- In many cases, the alternative hypothesis focuses on a particular direction

$$\boxed{\begin{array}{l} H_0\text{: } \mu \geq 3 \\ H_1\text{: } \mu < 3 \end{array}} \rightarrow$$

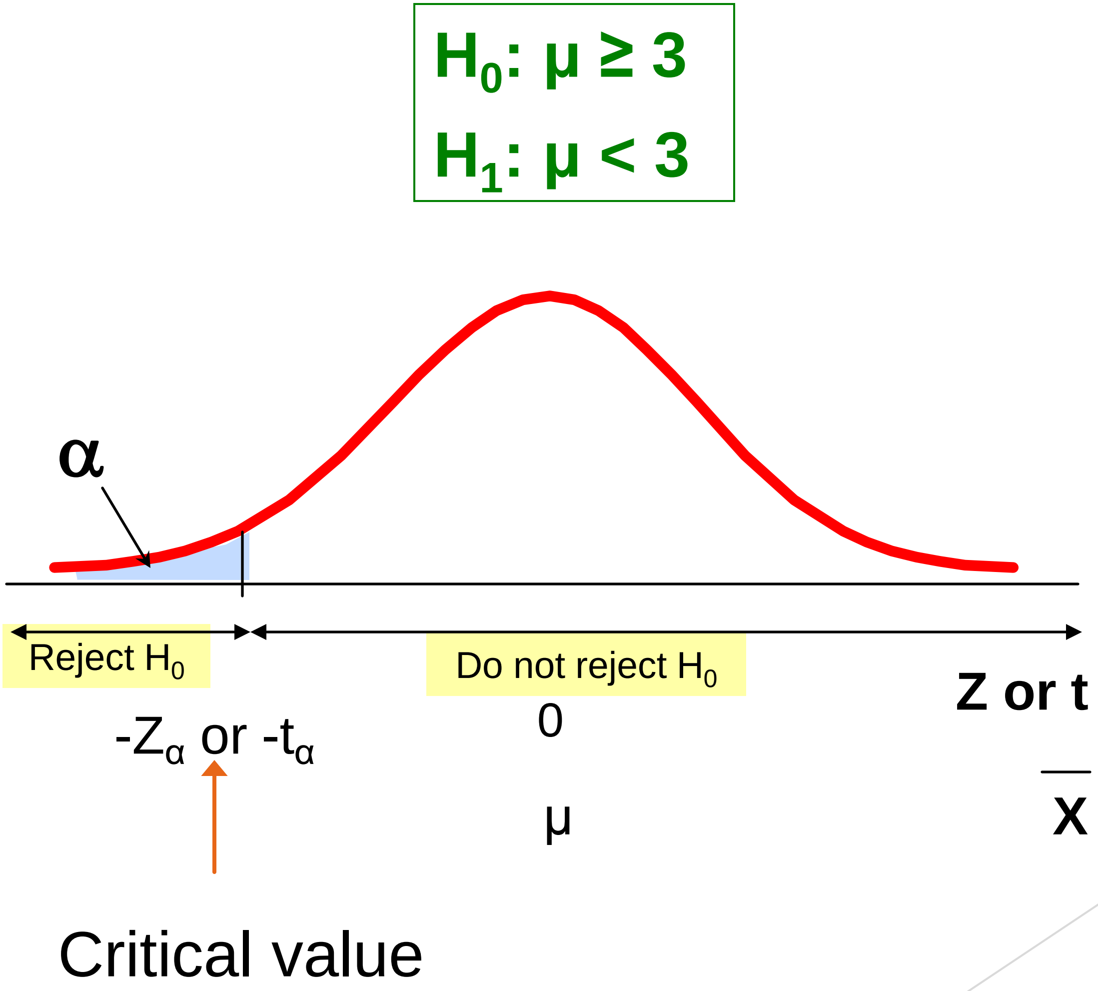
This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

$$\boxed{\begin{array}{l} H_0\text{: } \mu \leq 3 \\ H_1\text{: } \mu > 3 \end{array}} \rightarrow$$

This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

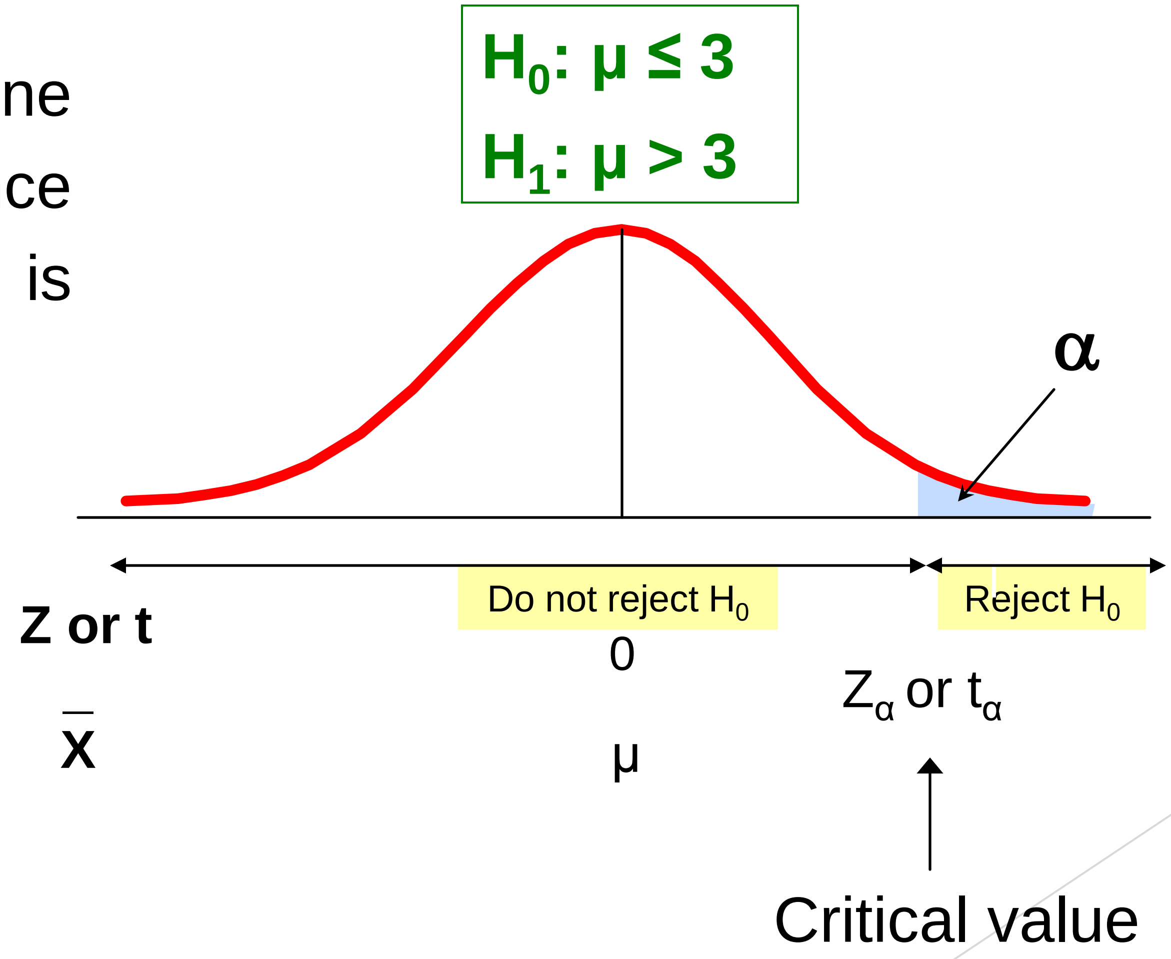
Lower-Tail Tests

- There is only one critical value, since the rejection area is in only one tail



Upper-Tail Tests

- There is only one critical value, since the rejection area is in only one tail



Example: Upper-Tail t Test for Mean (σ unknown)

A phone industry manager thinks that customer monthly cell phone bills have increased, and now average over \$52 per month. The company wishes to test this claim. (Assume a normal population)

Form hypothesis test:

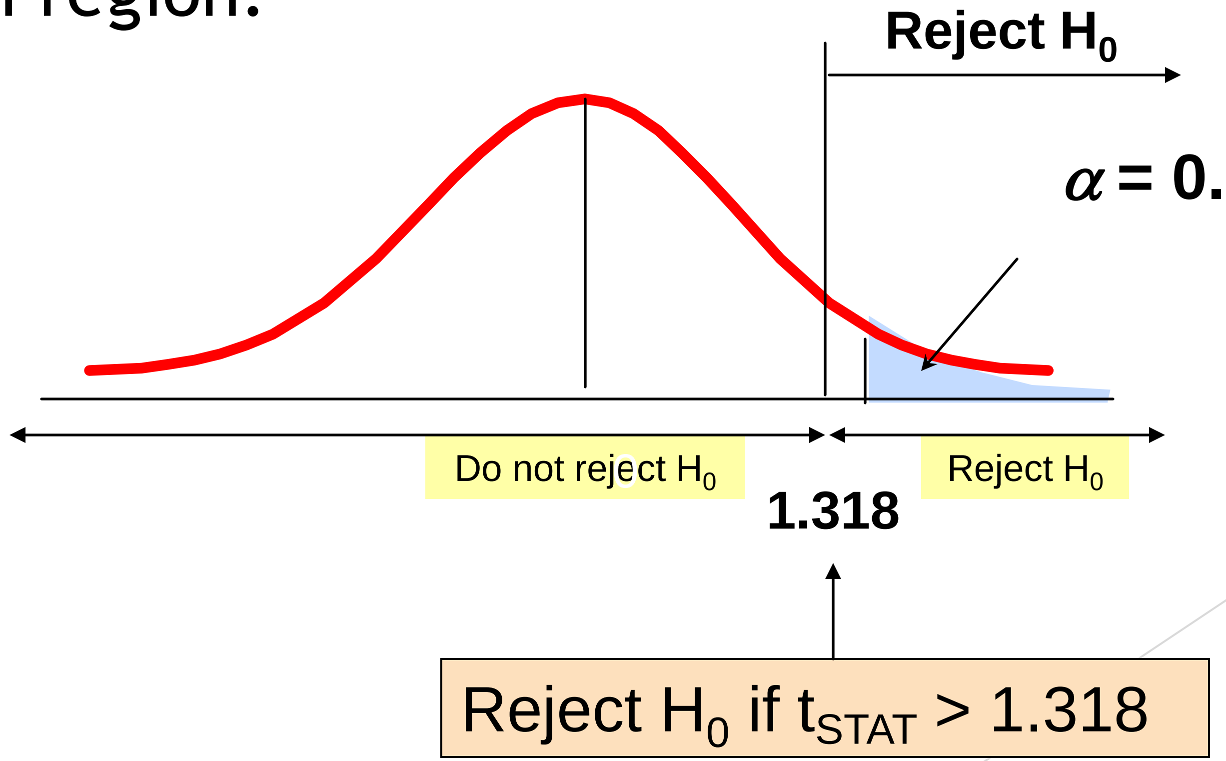
H_0: $\mu \leq 52$ the mean is not over \$52 per month

H_1: $\mu > 52$ the mean **is** greater than \$52 per month (i.e., sufficient evidence exists to support the manager's claim)

Example: Find Rejection Region

- Suppose that α = 0.10 is chosen for this test and n = 25.

Find the rejection region:



Example: Test Statistic

Obtain sample and compute the test statistic

Suppose a sample is taken with the following results:

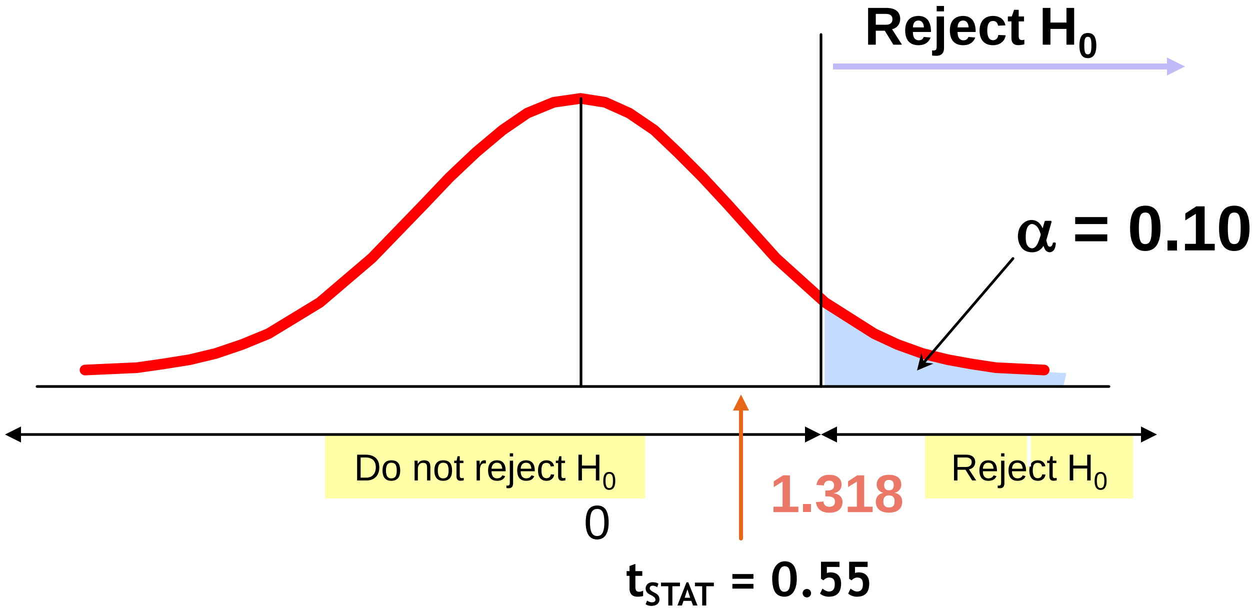
$n = 25$, $\bar{X} = 53.1$, and $S = 10$

- Then the test statistic is:

$$t_{STAT} = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{25}}} = \boxed{0.55}$$

Example: Decision

Reach a decision and interpret the result:

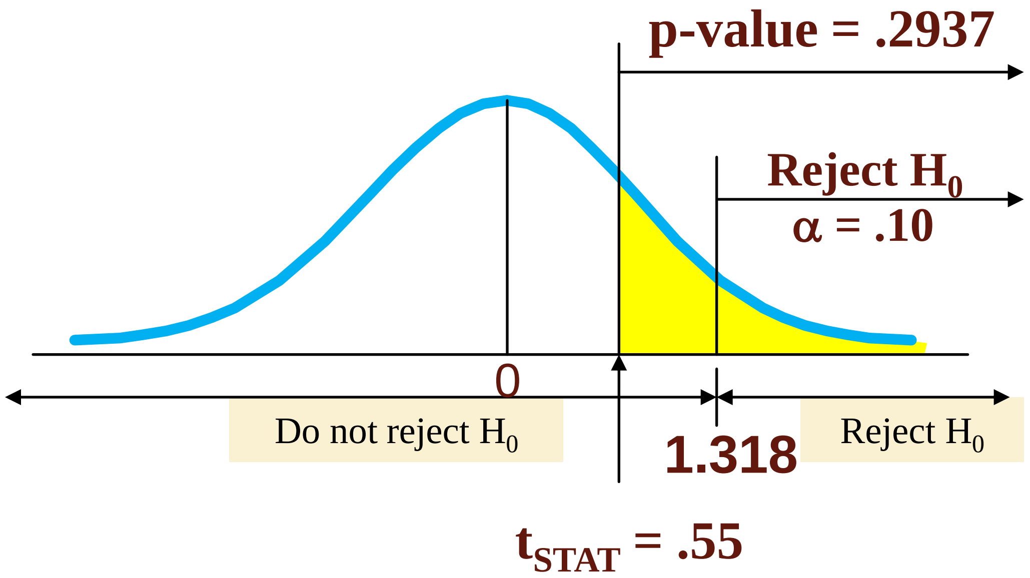


Do not reject H_0 since t_{STAT} = 0.55 < 1.318

there is not sufficient evidence that the mean bill is over $52

Example: Utilizing The p-value for The Test

- Calculate the p-value and compare to α



Do not reject H_0 since p-value = .2937 > α = .10